Caveats

- Problems in the slide will be independent from midterm problems
  - $P(p_1|p_1' \text{ in slide}) = P(p_1)$
- All numbers will be replaced by symbols in the slide
  - In midterm, you may need to compute
- We will cover top-K options from the Slido survey
  - Survey does not cover all topics
  - You still need to review all topics by yourself

Agenda & Survey result

- Generating RV with a specific distrition
- Function of RV
- Central Limit Theorem + Approx.
- BHT

Generating RVs

$U \sim \text{Uniform}[0, 1]$

- Goal: Given $F_X(c)$. Find $g(\cdot)$ s.t. $X = g(U)$ follows F_X

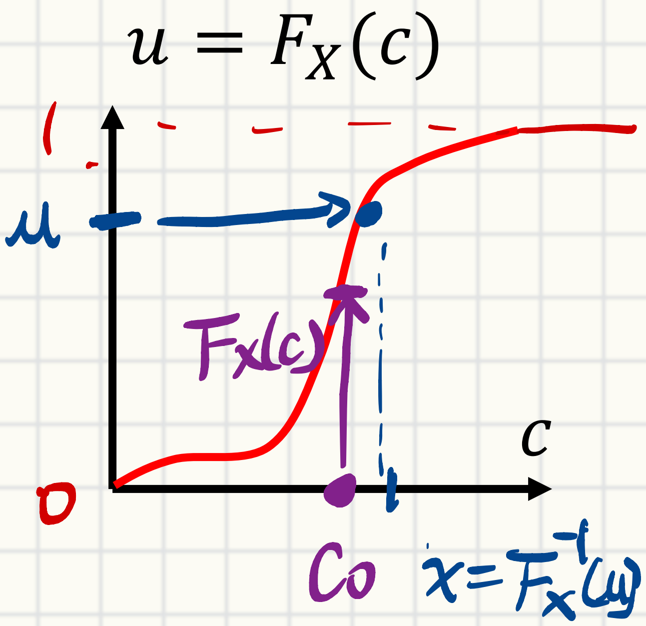
1. Start from F_X. Derive F_X if given f_X, p_X, F_X^c, etc.

2. $U = F_X(X)$ for any X

3. Let $u = f(x)$, find $g = f^{-1}(\overset{u}{\cancel{x}})$ s.t. $x = g\,(u)$

$g = F_x^{-1}(u)$

$x = F_x^{-1}(u)$



Example – Generate from PDF

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & else \end{cases}$$. Find $g(\cdot)$ s.t. $g(U)$ follows $f_X(\cdot)$

$$F_X(c) = \int_0^c 2e^{-2x}\,dx = \left[-e^{-2x}\right]_0^c = 1 - e^{-2x}$$

$$\underline{u = 1 - e^{-2x}}$$

$$\Rightarrow x = ?$$

$$e^{-2x} = 1 - u$$

$$-2x = \ln(1-u)$$

$$u = F_X(x)$$

$$x = -\frac{1}{2}\ln(1-u)$$

$$= \ln\left(\frac{1}{\sqrt{1-u}}\right)$$

$$g(u) = \ln\left(\frac{1}{\sqrt{1-u}}\right)$$

- More examples after functions of RV

Functions of RV

- Goal: Let $Y = g(X)$ and f_X/F_X , find f_Y/F_Y

1. Is Y continuous? What is the support? Write trivial regions first

2. $F_Y(c) = P(Y \leq c) = P\{g(X) \leq c\} = \int_{g(X)\leq c} f_X(u)du$

3. $f_Y(c) = F_Y{}'(c)$

$E[Y] = \int g(x) f_X(u) du,$

$E[Y] = \int_{g(x)\leq c} Y f_X(u)du$

Example – Piecewise function

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

$Y = 2|X| + 3$, $X \sim N(1,4)$, find $F_Y(c)$ in terms of Φ

1. Support $Y \Rightarrow [3, \infty)$

$$F_Y(c) = \begin{cases} 0 & \text{if } c<3 \\ & \text{else} \end{cases}$$

2. $F_Y(c) = P\{2|X|+3 \le c\}$

2a. $x \ge 0 \Rightarrow P\{2X+3 \le c\} = P\{0 \le X \le \frac{c-3}{2}\}$

2b. $x < 0 \Rightarrow P\{-2x+3 \le c\} = P\{\frac{3-c}{2} \le X < 0\}$

if we have f_X $\Rightarrow F_Y(c) = \int_{\frac{3-c}{2}}^{\frac{c-3}{2}} f_X(u)du.$

$$z = \frac{X - \mu_X}{\sigma_X} = \frac{X-1}{2}$$ $\Rightarrow$ Make regions above in terms of z.

$$P\left\{0 \le X \le \frac{c-3}{2}\right\} \xlongequal{\frac{X-1}{2}} P\left\{\frac{0-1}{2} \le \frac{X-1}{2} \le \frac{\frac{c-3}{2}-1}{2}\right\}$$

$$= P\left\{-\frac{1}{2} \le Z \le \frac{c-5}{4}\right\}$$

$$P\left\{\frac{3-c}{2} \le X < 0\right\} = P\left\{\frac{\frac{3-c}{2}-1}{2} \le \frac{X-1}{2} < \frac{0-1}{2}\right\}$$

$$= P\left\{\frac{1-c}{4} \le Z < -\frac{1}{2}\right\}$$

$$F_Y(c) = \int_{\frac{1-c}{4}}^{\frac{c-5}{4}} f_Z(u)\,du = \boxed{\Phi\left(\frac{c-5}{4}\right) - \Phi\left(\frac{1-c}{4}\right)}$$

Examples – Case by case

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $Y = \begin{cases} X & if\ x \ge 0 \\ X^2 & else \end{cases}$, where $X \sim Uniform[-1,1]$, find f_Y

$\text{Support}_Y = [0, 1]$ $\quad F_Y(c) \begin{cases} 0 & c<0 \\ & \text{if } c \ge 0 \end{cases}$

- $F_Y(c)$

$P\{0 \le X \le c\} + P\{-\sqrt{c} \le X < 0\} = \int_{-\sqrt{c}}^{c} f_X(u)du = \int_{-\sqrt{c}}^{c} \frac{1}{2} du$

- Let $Y = g(U)$, find g

$\Downarrow$

$P\{X^2 \le c\}$

$= P\{X \ge -\sqrt{c}\}$

$= \boxed{\frac{c+\sqrt{c}}{2}}$

$$u = \frac{c+\sqrt{c}}{2}$$

$$2u = c+\sqrt{c}$$

$$2u+\frac{1}{4} = c+\sqrt{c}+\frac{1}{4}$$

$$= \left(\sqrt{c}+\frac{1}{2}\right)^2$$

$\Rightarrow c = ?$

$$\sqrt{c}+\frac{1}{2} = \sqrt{2u+\frac{1}{4}} = \frac{1}{2}\sqrt{8u+1}$$

$$c = \frac{1}{4}\left(\sqrt{8u+1} - 1\right)^2$$

$$g(u) = \frac{1}{4}\left(\sqrt{8u+1} - 1\right)^2$$

$u=0 \quad y=0$

$u=1 \quad y=1$

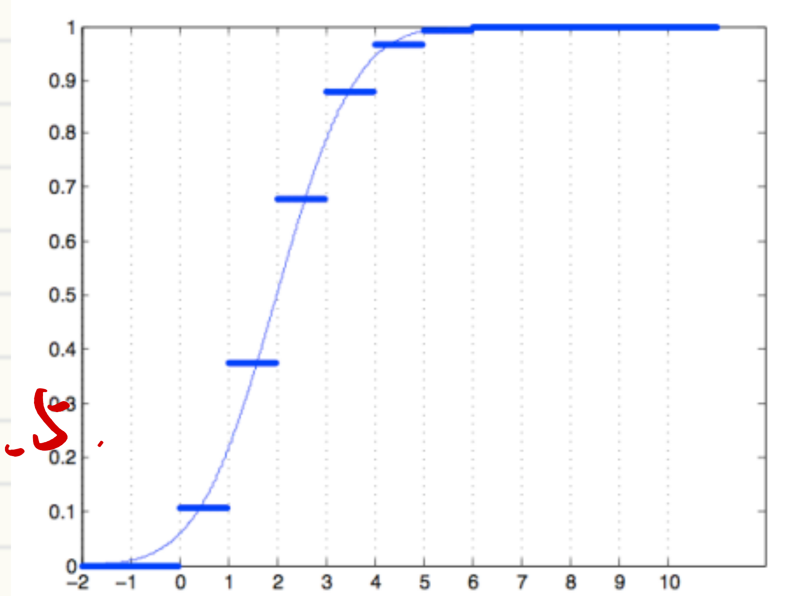
Central Limit Theorem (CLT)

If many independent random variables are added together, and if each of them is small in magnitude compared to the sum, then the sum X has an approximately Gaussian distribution $\tilde{X}$.

- $P\{X \leq v\} \approx P\{\tilde{X} \leq v\}$
- $\tilde{X} \sim N(\mu_X, \sigma_X^2)$

independent

- E.g. $X \sim Bi(100{,}0.5) + Bi(25, 0.9)$



* Beyond scope

$\mu_X \Rightarrow 100 \times 0.5 + 25 \times 0.9$

$= 50 + 22.5 = 72.5$

$\sigma_X^2 = 100 \times 0.5 \times 0.5 + 25 \times 0.9 \times 0.1$

$= 25 + 2.25 = 27.25$ $\tilde{X} \sim N(72.5, 27.25$

Gaussian Approximation

Discrete.

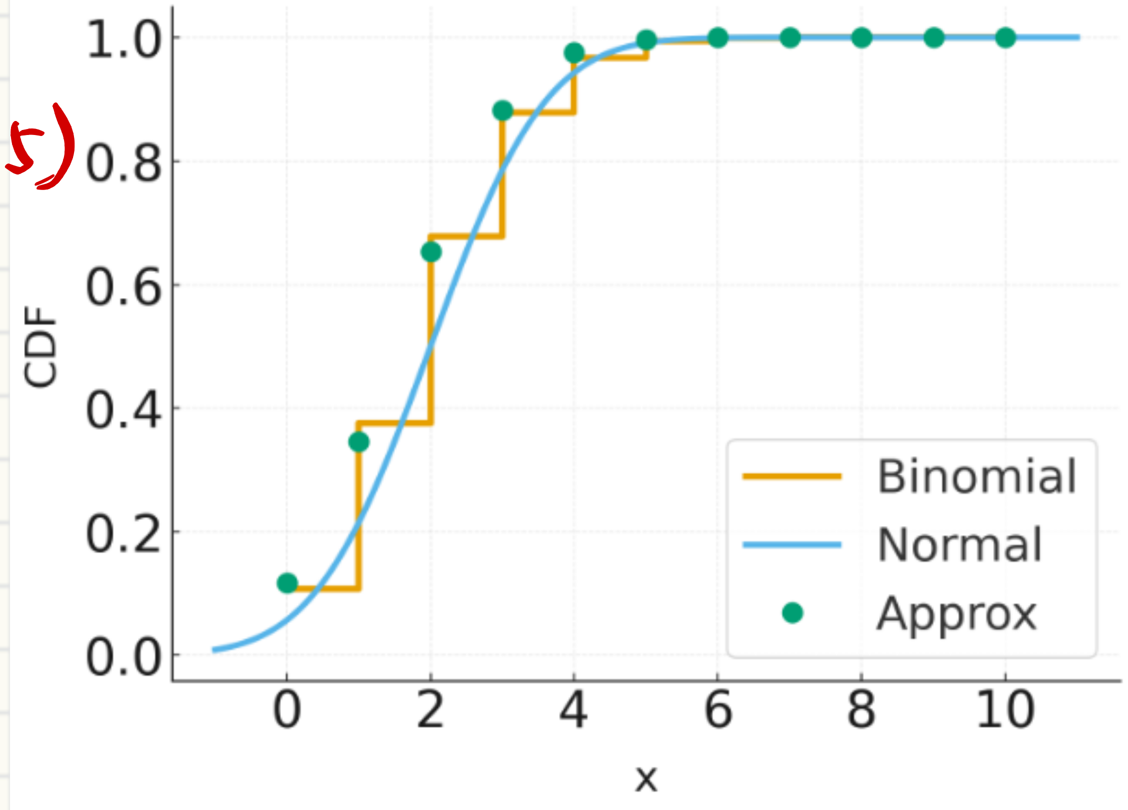
Approximate $X \sim Bin(10, 0.2)$ with $\tilde{X} \sim N(2, 1.6)$

- $F_X(2) = F_X(2.1) = F_X(2.9)$
- $F_{\tilde{X}}(2) \neq F_{\tilde{X}}(2.9)$
- How should we approximate?

cont. correction $\Rightarrow F_{\tilde{X}}(k \pm 0.5)$

- $P\{X \leq k\} \approx$
- $P\{X < k\} \approx$
- $P\{X \geq k\} \approx$
- $P\{X > k\} \approx$

$\rightarrow$ Priciple : (T/F) $k \in$ LHS $=$
(T/F) $k \in$ RHS



1. $P(X \le k) \approx \boxed{P(\tilde{X} \le k+0.5)} \,/\, P(\tilde{X} \le k-0.5)$

$\Rightarrow$ T/F

	$x=k$	$\tilde{x}=k$	$\tilde{x}=k$
	T	T	F

2 $P(X > k) \approx \boxed{P(\tilde{X} > k+0.5)} \,/\, P(\tilde{X} > k-0.5)$

	$x=k$	$\tilde{x}=k$	$\tilde{x}=k$
T/F	F	F	T

Example -

Consider a binary game, $P_X(2) = 0.9$ and $P_X(-2) = 0.1$. Play the game 100 times, denoted as $X_1 \dots X_{100}$

- Consider $Z_i = (X_i + 2)/4$, what distribution does Z_i follows?

$$\begin{cases} 90\% & Z_i = (2+2)/4 = 1 \\ 10\% & Z_i = (-2+2)/4 = 0 \end{cases}$$

$Z_i \sim \text{Bern}(p = 0.9)$

- Approximate $P\{X \geq 40\}$ in terms of Φ

$X \geq 40$

$Z_i = (X_i + 2)/4$

$X_i = 4Z_i - 2$

$$\sum_{i=1}^{100} Z_i \sim Bi(100, 0.9)$$

$\parallel$

Z

$\sigma_Z^2 = 100 \times 0.9 \times 0.1 = 9$

$$P\{\sum_{i=1}^{100} X_i = \sum_{i=1}^{100}(4Z_i - 2) = 4Z - 200 \geq 40\}$$

$= P\{Z \geq 60\}$ $\quad \tilde{Z} = N(90, 9)$

↳ corr. $(60 \geq 60) \Rightarrow (60 \geq 59.5)$

Binary Hypothesis Testing

$$P\left(\frac{Z-90}{3} \geq \frac{59.5-90}{3}\right)$$

$$= 1 - \Phi\left(\frac{-30.5}{3}\right)$$

- Likelihood matrix/ function
  - $f_i(k) = P\{X = k|H_i\}$
  - Likelihood Ratio $\Lambda(k) = \frac{f_1(k)}{f_0(k)}$
  - ML method - $f_1(k)\ vs.\ f_0(k)$ or $\Lambda(k) > 1$? Claim H_1

    <1 Claim H_0

- Joint Probability Matrix $P(X_i H_i)$
  - MAP method - $\Lambda(k) > \tau_{MAP} = \frac{\pi_0}{\pi_1}$

$Z \sim Bi(100, 0.9) \quad P\{Z \geq 60\}$

$\tilde{Z} \sim N(\mu_Z, \sigma_Z^2) = N(90, 100 \times 0.9 \times 0.1)$

$= N(90, 9)$

Z is discrete

$P\{Z \geq 60\} \approx ? \quad P\{\tilde{Z} \geq 60.5\} \, / \, \boxed{P\{\tilde{Z} \geq 59.5\}}$

$\Rightarrow$ check $\quad z = k = 60 \quad \tilde{Z} = 60 \quad \tilde{Z} = 60$

T $\quad$ F $\quad$ T

$= P\{\tilde{Z} \geq 59.5\} = P\left\{\frac{\tilde{Z} - 90}{3} \geq \frac{59.5 - 90}{3}\right\}$

$\boxed{\text{Exam Safe}}$ Discrete PMF approximation

$$P_X(X=k) = F_{\tilde{X}}(k+0.5) - F_{\tilde{X}}(k-0.5)$$

Example -

Let $f_1 \sim N(\mu = 1, \sigma^2 = 2)$ and $f_0 \sim N(\mu = 0, \sigma^2 = 1)$

- Find ML rule and MAP rule corresponding $P_{miss}, P_{false\ alarm}$

$$f_1(x) = \frac{1}{\sqrt{2\pi 2}} \exp\left(\frac{-(x-1)^2}{2\times 2}\right) \qquad f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

$$\Lambda = \frac{1}{\sqrt{2}} \exp\left(\frac{2x^2 - (x-1)^2}{4}\right) \overset{?}{>} 1$$

ML

MAP $> \frac{\pi_0}{\pi_1}$

$$\exp\left(\frac{2x^2-(x-1)^2}{4}\right) > \sqrt{2}$$

$$\Rightarrow x > \sqrt{2\ln 2+2} - 1 \quad \text{or}$$

$$x < -\sqrt{2\ln 2+2} - 1$$

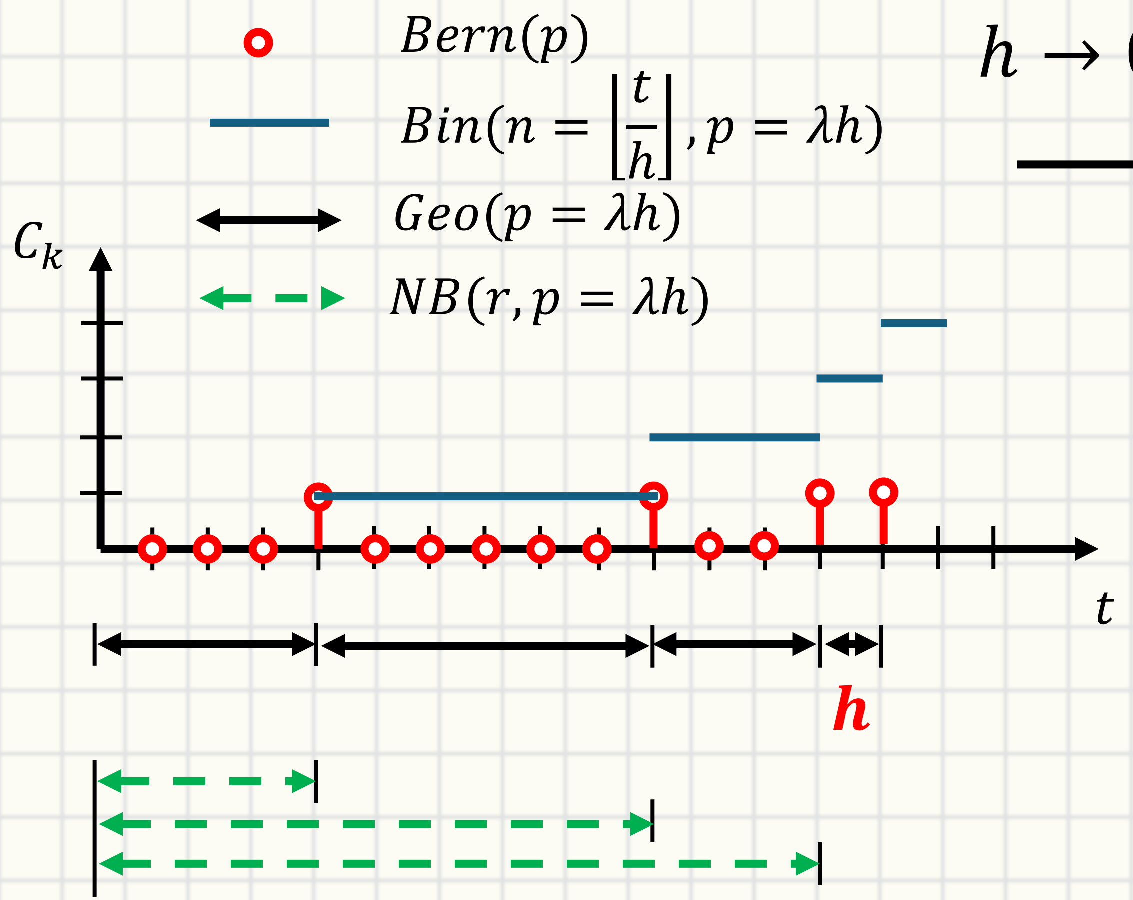
P_{false}

$$P\{\text{Claim } H_1 \mid H_0\} \Rightarrow f$$

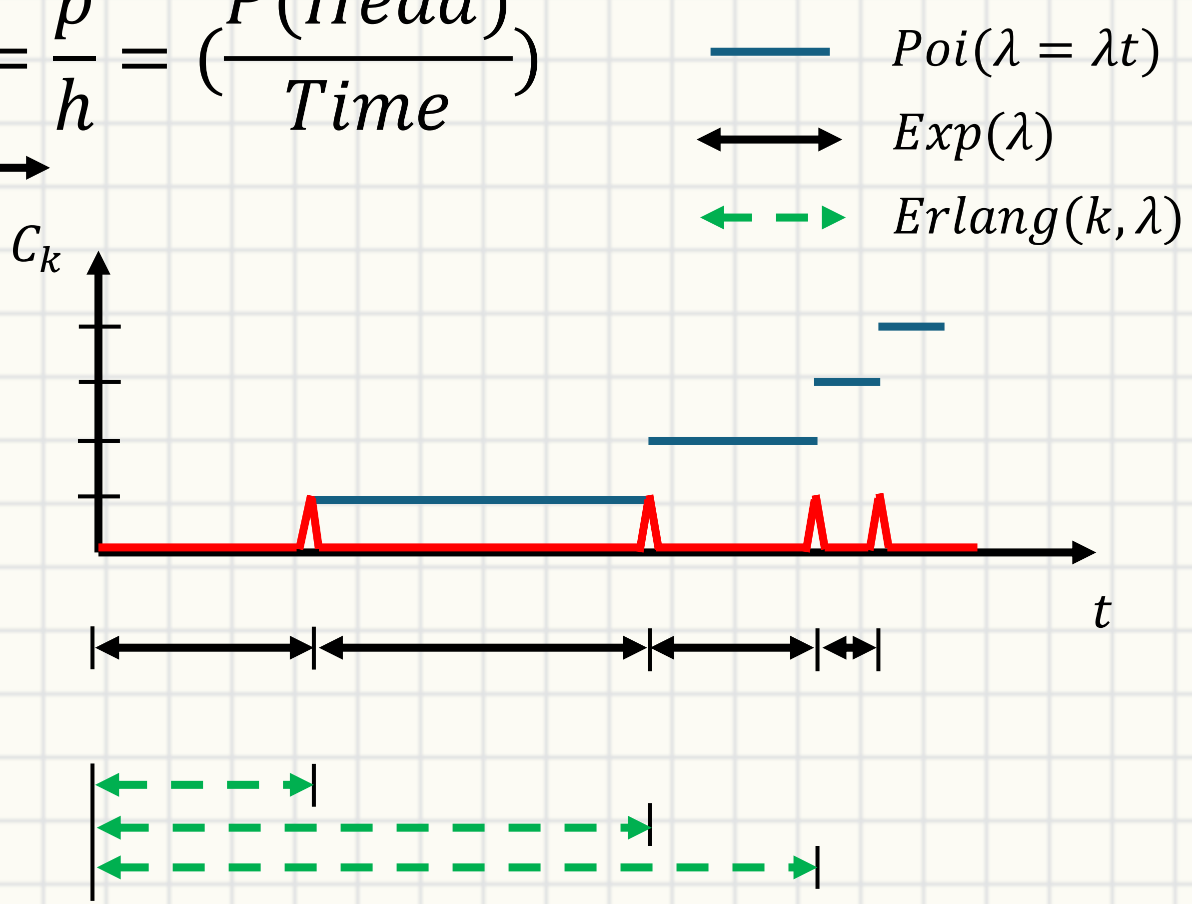
$$= \Phi\left(-\sqrt{2\ln 2+2} - 1\right) + Q\left(\sqrt{2\ln 2+2} - 1\right)$$

Bernoulli Process $\xrightarrow{h \to 0, \lambda = \frac{p}{h}}$ Poisson Process

- Assume each trial takes h duration to complete



$$h \to 0, \lambda = \frac{p}{h} = \left(\frac{P(Head)}{Time}\right)$$



Properties

	$Exp(\lambda)$	$Poi(\lambda = \lambda_{ref}t)$
Mean		
Variance		
PDF/ PMF		
CDF		
Example	System lifetime	Event occurrence within t
Special		