Last lecture

Distributions

- Bernoulli (Ch 2.4.3)
- Binomial (Ch 2.4.4)

$X \sim Ber(p)$

$p_X(1) = p \quad \mu_X = p$

$\sigma_X^2 = p(1-p)$

$Y \sim Bi(n,p)$

$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\mu_Y = np \quad \sigma_Y^2 = np(1-p)$

Geometric distribution (Ch 2.5)

- Definition, mean and variance

$L \sim Geo(p) \quad p_L(k) = (1-p)^{k-1} p$

$\mu_L = \frac{1}{p} \quad Var(L) = \frac{1-p}{p^2}$

Agenda

Correction on Best-of-K

Geometric distribution (Ch 2.5)

- Memoryless property

Bernoulli Process (Ch 2.6)

- Definition
- Connect to distributions

Poisson Distribution (Ch 2.7)

Maximum Likelihood Estimation (MLE)

Binomial Example – Best of K

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Team A and B play "Best of 7" games

- No tie, whoever wins 4 games out of 7 is the match winner
- E.g. $w_i = \{A, A, A, B, A\}$: the winner is A
- Let p denotes A's win rate per game
- Y denotes the number of games played, $p_Y(k) = ?$

$P_Y(0) = P_Y(1) = P_Y(2) = P_Y(3) = 0$

$k \geq 4$

$P_Y(k) = P_Y(k, A) + P_Y(k, B)$

$P_Y(k, A) = \binom{k-1}{4-1} p^4 (1-p)^{k-4}$

Last game must be win by A

k

A

k-1

Geometric Distributions

Property – Memoryless property

For geometric series, failing 10 times will not affect the 11-th trial

↱ Already get n "T"

- $P\{L > k + n | L > n\} =$ $P\{L > k\}$

↳ At least k+n "T"

- Called "memoryless property"
- What's the expected total number to get the first 1 after getting {0,0,0,0}?

$L \sim Geo(p)$

↓

$4 + \mu_L = 4 + \frac{1}{p}$

Game – Push the luck (simplified Incan-Gold)

Start a game with infinite rounds and 0 points

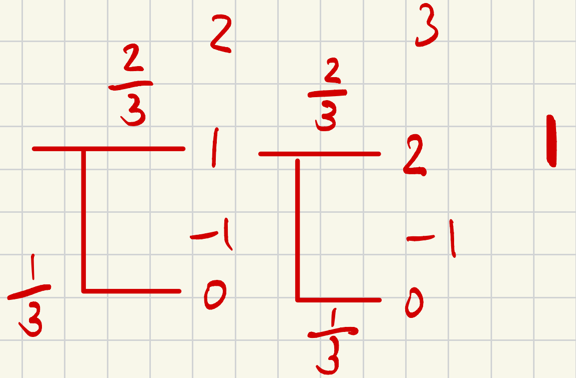
- 2 Actions per round – Go or Keep
- Go
  - $p = \frac{2}{3}$ win 1 point
  - Otherwise, lose all points
- Keep
  - Deposit the current point and end the game
- What's the best strategy?

Go until round k^* k^*?

① $L \sim Geo(\frac{1}{3})$ $\mu_L = 3$. $k^* = \mu_L - 1 = 2$

② Go @ 3rd round $k^* = 3$

<1> Expected outcome per round $E_S[G] = (\frac{2}{3})^k k$



$\frac{2}{3} < \frac{8}{9} = \frac{24}{27} > \downarrow$ - - - -

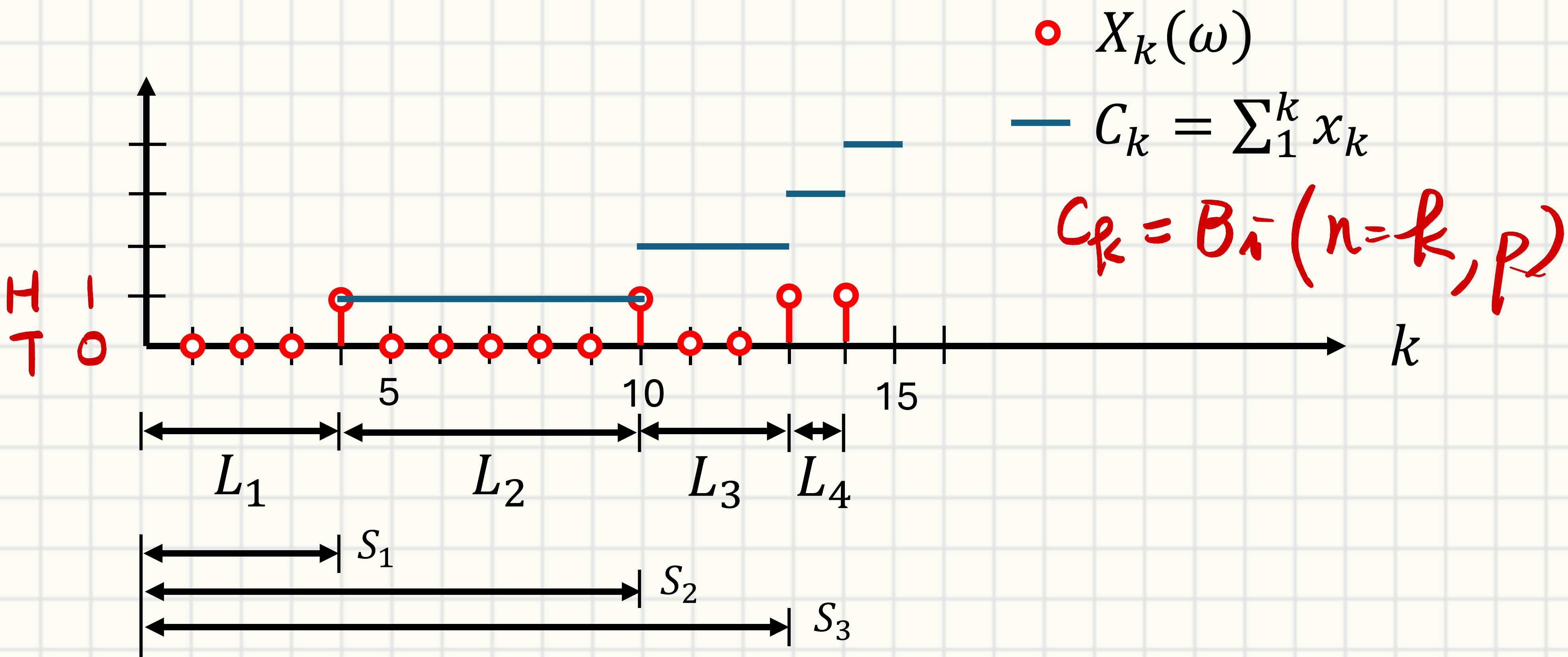
$k = 1 \quad 2 \quad 3$

Bernoulli Process

Bernoulli Process Definition

An infinite sequence X_1, X_2 … s.t. $X_k \sim Bern(p)$

- ω is a possible outcome of the sequence
- $X_k(\omega)$ is called a " realization " of outcome ω



Bernoulli Process Definition

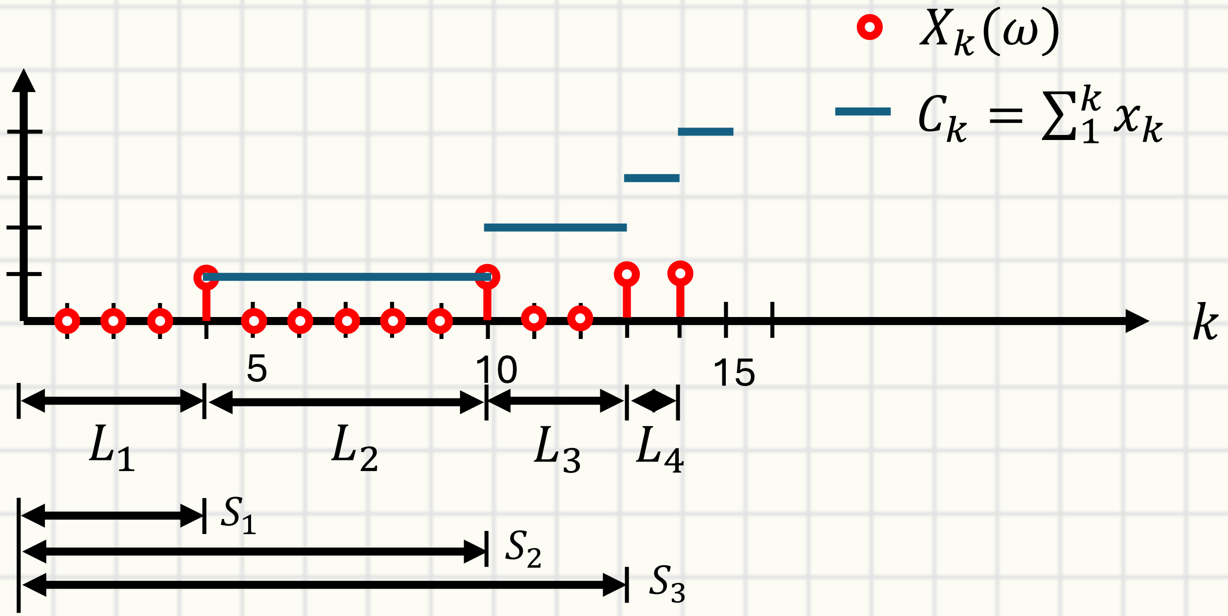
Observe that a Bernoulli process can be defined by

1. $X_k \sim Bern(p)$
2. $C_k \sim \sout{B(k,p)}$ Bi(k, p)
3. $L_k \sim \sout{L(p)}$ Geo(p)
4. $S_r = \sum_1^r L_r$: # of trials required to get r ones

⇓

Best-of-K games

$Y_A = S_4$

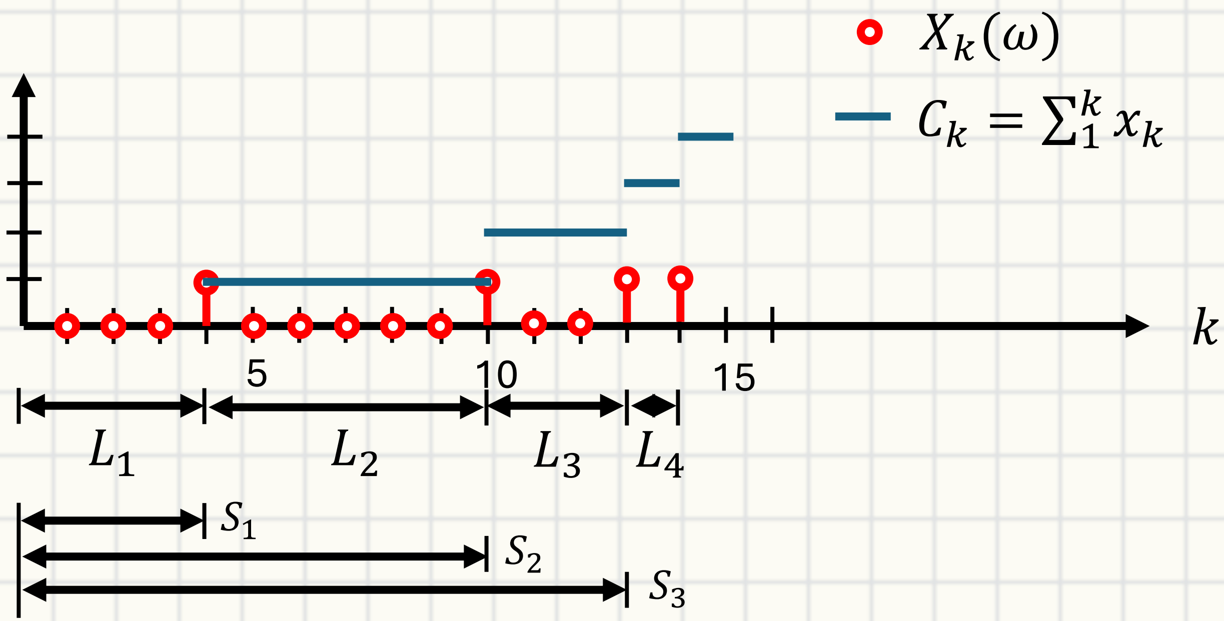


S_r - Negative Binomial Distribution

What is the pmf of S_r with parameter (r, p)?

- # of trials required to get r ones
- $p_S(n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$

↳ last toss must be 1



Poisson Distribution

Poisson Distribution $Pois(\lambda)$

A binomial distribution with large n, small p, and $\lambda = np$

- Example – Misspelled words in a document
  - Many number of words n
  - Small misspelled rate p

$\Rightarrow Bi(1000, 0.01)$

$\sim Bi(10000, 0.001)$

$\sim Bi(\frac{10}{p}, p)$

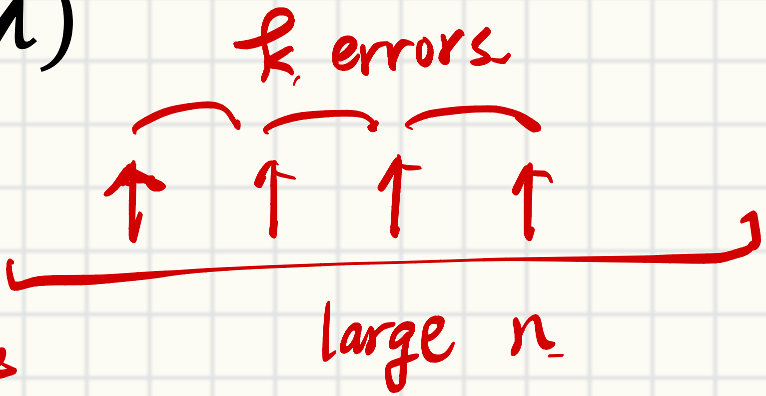
- When we care about the “rate” np
  - We only have the mean np
  - We know p is small

- $p_X(k) = \frac{e^{-\lambda}\lambda^{k}}{k!}$

Poisson Distribution $Pois(\lambda)$

- Why $p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$?
  - $p_X(k) \propto \frac{\lambda^k}{k!}$ (λ → failing rate, $k!$ → permute)
  - $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\mu_x = \lambda$
- $\sigma_x^2 = \lambda$

$\sigma_Y^2 = np(1-p) = \lambda$



$\lambda = np$. $Y \sim Bi(n, p)$

$P_X(k) \sim P_Y(k)$

$= \binom{n}{k} p^k (1-p)^{n-k}$

$= \frac{n!}{k!(n-k)!} p^k$

Poisson Distribution Example

Consider a wireless link features bit error rate $1e^{-4}$

- $X \triangleq$ # of error bits in 1 Byte, $P_X(0) = ?$

$X \sim Bi(8, 0.0001)$ $\quad P_X(0) = \binom{8}{0}(0.0001)^0(0.9999)^8$

$= (0.9999)^8$

~ 0.9992

- $Y \triangleq$ # of error bits in 10MB , $P_Y(10) = ?$

$Y \sim Bi(8 \times 10^7, 10^{-4})$

$\sim Poi(\lambda = 8 \times 10^3)$

$P_Y(10) = \frac{e^{-\lambda} \lambda^{10}}{10!}$

Poisson Distribution Example

Consider a wireless link features bit error rate $1e^{-4}$

- Each packet (N bits) is governed by an error correction algorithm that can correct up to 2 bits
- If we want packet error rate lower than 0.01, what's the maximum packet size N?

- $P_X(0) + P_X(1) + P_X(2) \geq 0.99$
- $e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) \geq 0.99$
- $\lambda = Np \leq 0.436$
- $N \leq 4360$