

# Last lecture

## Law of Total Probability ([Ch 2.10](#))

- Bayes formula

## Independent Events/ RVs ([Ch 2.4](#))

- Definition
- Motivation
- Examples and Facts

# Agenda

## Distributions

- Bernoulli (Ch 2.4.3)
- Binomial (Ch 2.4.4)

Geometric distribution (Ch 2.5)

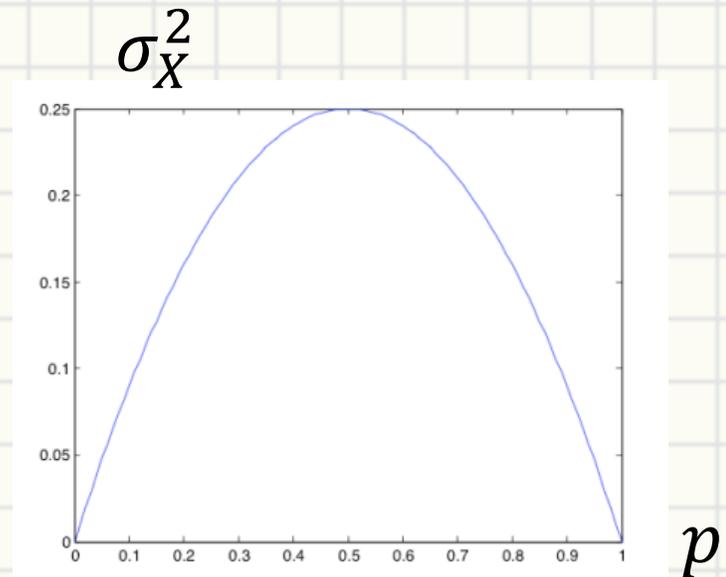
# Distributions

Common discrete RVs

# Bernoulli Distribution

$X$  is Bernoulli distribution with parameter  $p$  if

- $P\{X = 1\} = p$  and  $P\{X = 0\} = 1 - p$
- “Toss a (unfair) coin with  $p$  probability Head”
- Only two possible outcomes, pmf contains two bins
- $E[X] =$
- $E[X^2] =$
- $\sigma_X^2 =$



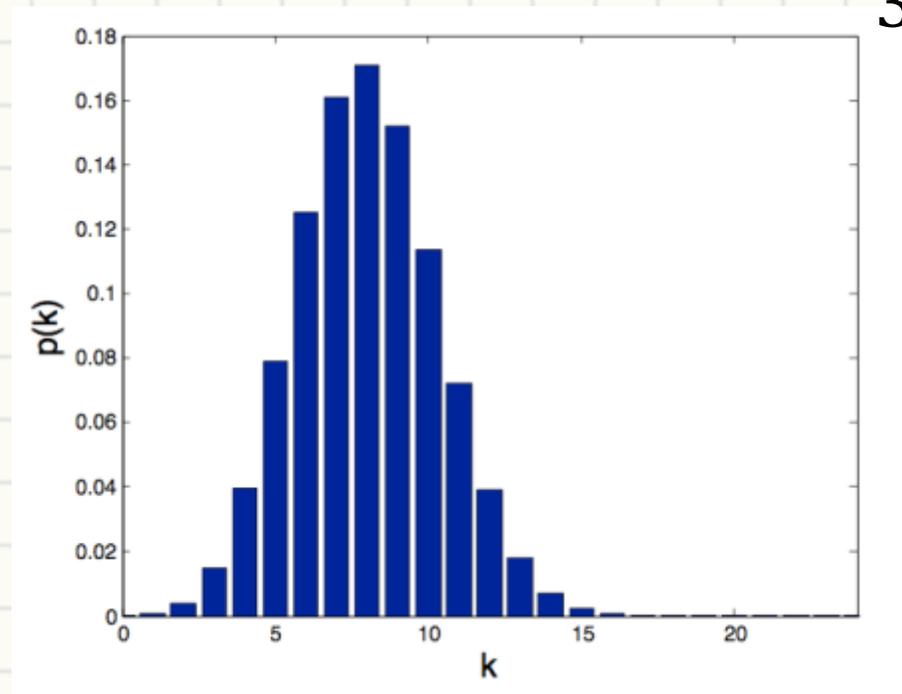
# Binomial Distribution

$X$  is binomial distribution with parameter  $(n, p)$  if

- $X$  is sum of  $n$  Bernoulli trials with parameter  $p$
- Draw the unfair coin  $n$  times and count the Head

$$(n, p) = \left(24, \frac{1}{3}\right)$$

- $p_X(k) =$



# Binomial Distribution

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Proof -  $\sum_{k=0}^n p_X(k) = 1$

- $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$
- $x = \frac{p}{1-p}$

# Binomial mean

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^n k \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^n \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{m=0}^{n-1} \frac{n-1!}{(n-1-m)!m!} p^m (1-p)^{n-1-m}$$

$$= np \sum_{m=0}^j \binom{j}{m} p^m (1-p)^{j-m}$$

# Binomial Properties

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Mean  $E[X] =$
- Variance  $Var(X) =$ 
  - why?
- Shape of the pmf- what is the most likely  $k$ ?
  - $k^* = \lfloor (n+1)p \rfloor$

# Slido

Back to early 1900's, there's a mail fraud

- I mail  $N$  people that I can predict a series of 50-50 games
- I predict  $A$  wins in  $\frac{N}{2}$  mail,  $B$  wins for the other
- Stop mailing a person after 2 wrong guess
- Say people will subscribe to my prediction if I only make at most 1 mistake in 5 guesses
- What should be  $N$  if I want to get 300 subs?



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$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

# Binomial Example – Best of K

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Team A and B play “Best of 7” games

- No tie, whoever wins 4 games out of 7 is the match winner
- E.g.  $w_i = \{A, A, A, B, A\}$ : the winner is A
- Let  $p$  denotes A's win rate per game
- $Y$  denotes the number of games played,  $p_Y(k) = ?$

# Geometric Distributions

# Geometric Distribution

# of Toss on a (unfair) coin until the first Head is shown

Conduct independent Bernoulli trials of parameter  $p$

- $L \triangleq$  # of trials until we get the first 1
- $p_L(1) =$
- $p_L(2) =$
  
- $p_L(k) =$
  
- $P\{L > k\} =$

# Mean

$$E[L] =$$

# Variance

$$\text{Var}(L) =$$

# Example

What's the expected number of rolls to get 1 to 6 at least once?

- Example series :  
 $\{2, 4, 2, 3, 4, 4, 3, 5, 3, 5, 4, 4, 6, 2, 3, 3, 4, 1\}$
- $R_k \triangleq$  # roll to get the  $k$ -th unseen number

# Property – Memoryless property

For geometric series, failing 10 times will not affect the 11-th trial

- $P\{L > k + n | L > n\} =$
- Called “memoryless property”
- What’s the expected total number to get the first 1 after getting {0,0,0,0}?

# Game – Push the luck (simplified Incan-Gold)

Start a game with infinite rounds and 0 points

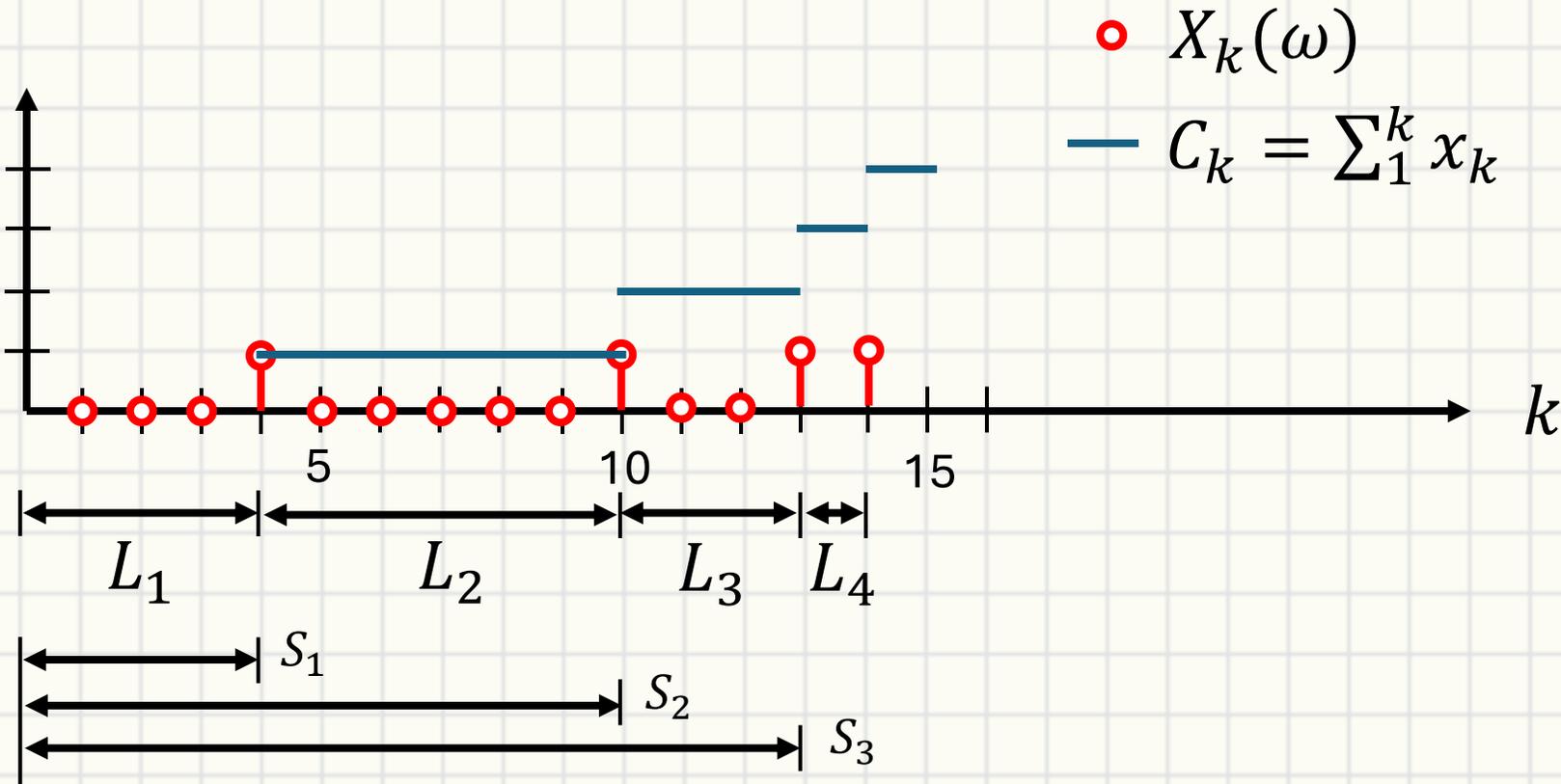
- 2 Actions per round – Go or Keep
- Go
  - $p = \frac{2}{3}$  win 1 point
  - Otherwise, lose all points
- Keep
  - Deposit the current point and end the game
- What's the best strategy?

# Bernoulli Process

# Bernoulli Process Definition

An infinite sequence  $X_1, X_2 \dots$  s.t.  $X_k \sim \text{Bern}(p)$

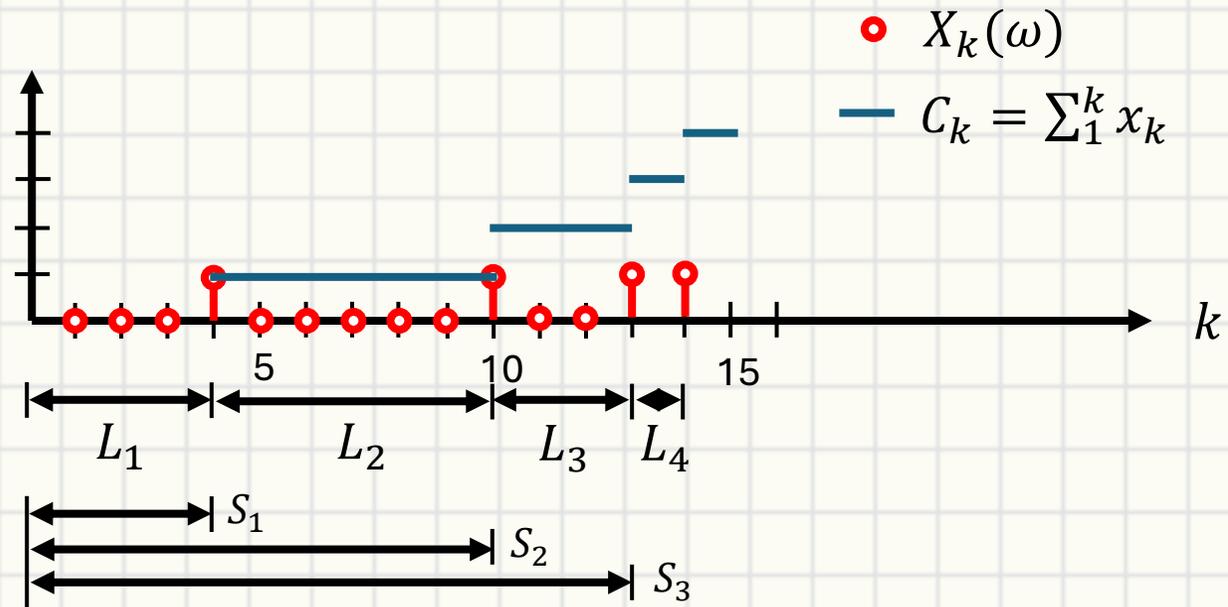
- $\omega$  is a possible outcome of the sequence
- $X_k(\omega)$  is called a “ ” of outcome  $\omega$



# Bernoulli Process Definition

Observe that a Bernoulli process can be defined by

1.  $X_k \sim \text{Bern}(p)$
2.  $C_k \sim B(k, p)$
3.  $L_k \sim L(p)$
4.  $S_r = \sum_1^r L_r$  : # of trials required to get  $r$  ones



# $S_r$ - Negative Binomial Distribution

What is the pmf of  $S_r$  with parameter  $(r, p)$ ?

- # of trials required to get  $r$  ones
- $p_S(n) =$

