Last lecture

Random Variables (RV) (Ch 2.1)

- Probability Mass Function (pmf)
- Mean and Variance (Ch 2.2)

Agenda

Random Variables (RV) (Ch 2.1)
- Mean and Variance recap (Ch 2.2)

Conditional Probability (Ch 2.3)
- Motivation
- Examples
- Solver

Law of Total Probability (Ch 2.10)

Mean and Variance

Mean and Variance of linear RV functions

Always true:

By LOTUS: // $\sum_i (ax_i + b)\, p(x_i) = a \sum_i x_i p(x_i) + b$

1. $E[aX + b] =$ $a\mu_x + b = aE[X] + b$
2. $E[X + Y] =$ $\mu_x + \mu_y$

By definition:

1. $Var(aX + b) =$ $a^2\, Var(X)$
2. σ_{aX+b} $= a\,\sigma_x$

$Var(X) = E[(X - \mu_x)^2]$
$= E[X^2] - \mu_x^2$

Detailed later - Requires the relationship between X, Y

1. $Var(X + Y)$ $= E[(X + Y - \mu_x - \mu_Y)^2]$
2. $E[(X + Y)^2]$

- All functions related to cross terms, e.g. $E[XY]$

Examples

$X \triangleq$ Roll a die

$A \in [2, 12]$

$A \triangleq$ "Rolling a die twice and sum the results"
$B \triangleq$ "Roll the same die once, multiply the result by 2"

$B \in \{2, 4, 6, 8, 10, 12\}$

- $p_A(k) = p_B(k)$? ✗
  possible outcomes are diff
- $E[A] = E[B]$? ✓
  $\mu_A = E[X_1 + X_2] = \mu_{X_1} + \mu_{X_2} = 2\mu_X$
  $\mu_B = E[2X_1] = 2\mu_X$
- $Var(A) = Var(B)$? ✗
  $Var(A) = Var(X_1 + X_2) \overset{\text{indep.}}{=} Var(X_1) + Var(X_2) = 2Var(X)$
  not in range now
  $Var(B) = Var(2X_1) = 4Var(X)$

Standardized RV

$\mu_Z = 0 \quad Var(Z) = 1 = \sigma_Z^2$

For any RV X

- $\frac{X-\mu_X}{\sigma_X}$ is a standardized RV – Mean 0, variance 1

$E[X - \mu_x] = \mu_x - \mu_x = 0$

$Var\left(\frac{X-\mu_x}{\sigma_x}\right) = \left(\frac{1}{\sigma_x}\right)^2 Var(X) = \frac{\sigma_x^2}{\sigma_x^2} = 1$

Conditional Probability (Ch 2.3)

Motivation

The probability of B happens given A happens

- $P($ pair of socks are same color $)$ given $S_1 = B$
- $P(I\ win\ Texas\ Hold'em)$ given $X = Ace + Ace$
- $P(I\ pass\ 313)$ given I skip HW1…

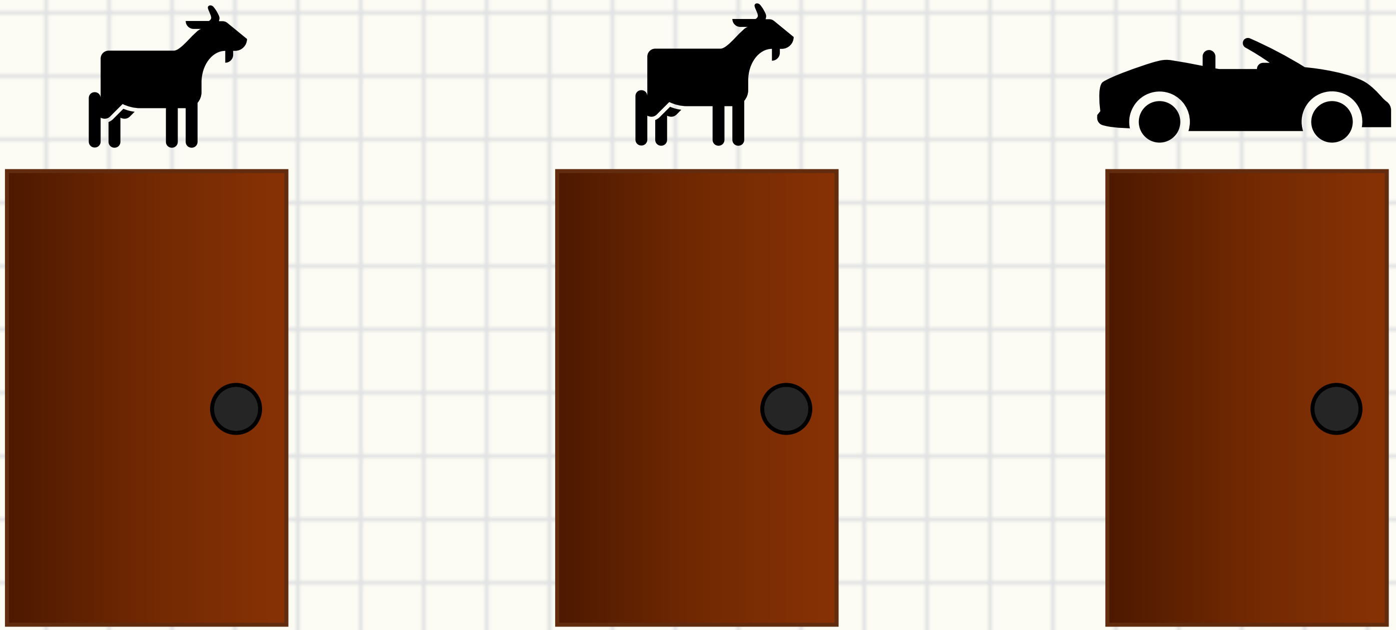
Why do we need conditional probability?

- Analyze the relationship between two events
- Find the optimal solution to make an event probable

Examples

3 doors (Monty Hall) problem

- 3 closed doors – 1 leads to a car, the others lead to goats
- After you choose one, the host will open a "Goat" door
- Should you change the door?



$P(X|Y) = P(X)$ conditioned on Y

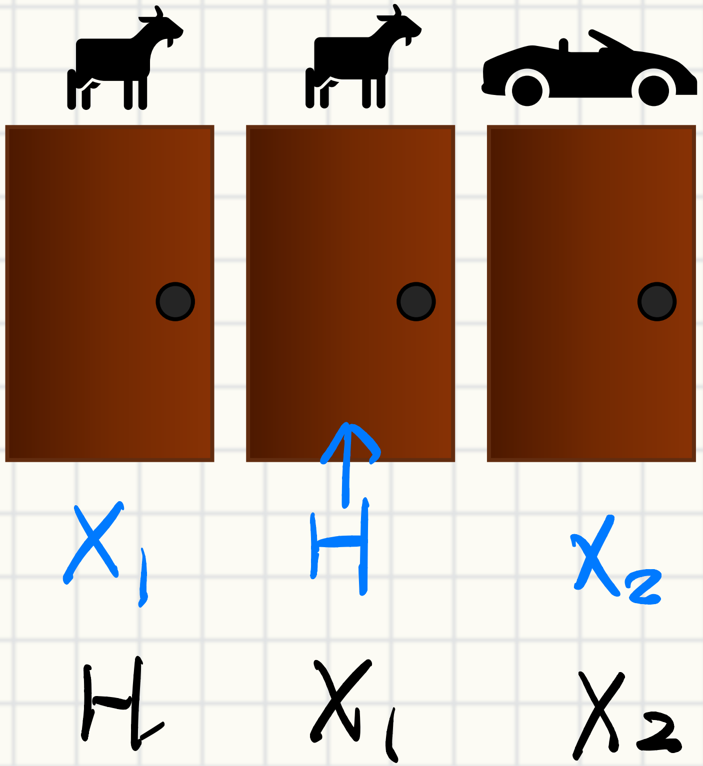
Examples

$X_i \in \{1, 2, 3\}$

- Let X_i denotes i-th choice
- Never change (NC)
  - $P(W|NC) = P(W|X_2 = X_1)$

    $= P(X_1 = Car) = \frac{1}{3}$
- Change (C)
  - $P(W|C) = P(W|X_2 \neq X_1)$

    $= P(X_1 = Goat)$?
- What if there are 4 doors… 2 cars and 2 goats?



Conditional Probability

$$P(B|A) = \begin{cases} \frac{P(AB)}{P(A)} & \text{if } P(A) > 0 \\ \text{Undefined} & \text{else} \end{cases}$$

Roll two dice, A = sum is 6; B = numbers are not equal

$P(B) =?$ $P(B|A) =?$ $P(B^c|A) =?$

$E_A = \{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\}$

$P(B) = \frac{5}{6}$

$P(B|A) = \frac{\frac{4}{36}}{\frac{5}{36}} = \frac{4}{5}$

$P(B^c|A) = \frac{\frac{1}{36}}{\frac{5}{36}} = \frac{1}{5}$

$E_{AB^c} = \{(3,3)\}$

Problem Solving

$P(A|B) \Rightarrow$ Is AB seperable from B?

CARDS

YES

$P($FULLHOUSE $|$ 3 OF A KIND$)$

$C_1 : C_5$. $\quad C_1 C_2 C_3$

↳ $P(A|B)$ directly.

NO, $P($FULLHOUSE $|$ $C_1 C_2$ are diff suits$)$

↳ $\frac{P(AB)}{P(B)}$

Facts of conditional probability

- $P(B|A) \geq 0$
- $P(B|A) + P(B^c|A) = 1$
- $P(\Omega|A) = 1$
- $P(AB) = P(A|B)P(B)$ ⇒ $P(A|B) = \frac{P(AB)}{P(B)}$
- $P(ABC) = P(A|BC)P(B|C)P(C)$

$BC = X$

$P(AX) = P(A|X)P(X)$

$P(AB) = P(A|BC)P(BC)$ → $P(B|C)P(C)$

Unfair Games

Not textbook

Alice and Bob are playing a game

- They both play one of "O" or "X" simultaneously
- Alice earns the profits as follows

	O	X
O	-1	2
X	2	-3

- If Alice wants to maximize her profits, $P_A(O) =$

⇒ Nash Equilibrium.

$P_A(O) = p$

	A O	A X
B O	$p(-1)$	$(1-p)(2)$
B X	$p(2)$	$(1-p)(-3)$

$E[A \mid B = \text{"O"}]$
$= -p + 2(1-p)$
$= 2 - 3p$

$$E[A \mid B = \text{"X"}] = 2p + (1-p)(-3)$$

$$= 5p - 3.$$

$$2 - 3p = 5p - 3$$

$$\Rightarrow p^* = \frac{5}{8}.$$

Nash Eq.

$$\min(2 - 3p, 5p - 3)$$

Law of total probability

$$2 - 3p^* = \frac{3}{8}$$

$$= 5p^* - 3$$