Last lecture

Minimum mean square error estimation (Ch 4.9)

- Recap
  - Constant estimators
  - Unconstrained estimators
  - Linear estimators
- Examples

Joint Gaussian Distribution (Ch 4.11)

- Motivation
- Facts
- Examples

Agenda

- Law of Large Numbers and Central Limit Theorem (Ch 4.10)
  - Equations
  - Examples

- Expectation Maximization (Not tested)
  - Motivation
  - Application
  - Intuition
  - Algorithm
  - Visualization

Equations – Law of Large Numbers

$X_1 \ldots X_N$ are uncorrelated, and $\mu_{X_k} = \mu$ and $\sigma^2_{X_k} \le C$

- Let $S_n = \sum_1^n X_n$, we have $P\left\{\left|\frac{S_n}{n} - \mu\right| \ge \delta\right\} \le \frac{C}{n\delta^2}$ for any δ
- Prove by Chebyshev

↑

average

↓

$$P\{|X-\mu| \ge a\} \le \frac{\sigma^2}{a^2}$$

Treat $\frac{S_n}{n} = A_n$ as RV

$$\mu_{A_n} = E\left[\frac{S_n}{n}\right] = \frac{\mu_{S_n}}{n} = \frac{E[\Sigma X]}{n} = \frac{\Sigma E[X]}{n} = \frac{n\mu}{n}$$

$$= \mu$$

$$\sigma^2_{A_n} = Var\left(\frac{\Sigma X_i}{n}\right) = \frac{1}{n^2} Cov(\Sigma X_n, \Sigma X_n)$$

$$\underset{\text{uncorr}}{=} \frac{1}{n^2}\Sigma\, Var(X_n) \le \frac{1}{n^2}\Sigma\, C = \frac{C}{n}$$

Example

$X_1 \dots X_{100}$ are RVs with $\mu = 5$ an $\underline{\sigma^2 = 1}$. $S_n = \sum_1^n X_n$

- Assume $|Cov(X_i, X_j)| \leq 0.1$ if $i = j \pm 1$
- $Cov(X_i, X_j) = 0$ if $|i - j| \geq 2$

- Show $Var(S_{100}) < 120$

$a=0.5$ Chebyshev

↓

- Find upper bound of $P\left(\left|\frac{S_{100}}{100} - 5\right| \geq 0.5\right)$

$$Var(S_{100}) = Cov(X_1 + X_2 \dots + X_{100}, X_1 + X_2 \dots + X_{100})$$

$$= \sum_{i=1}^{100} \underline{\sigma_{X_i}^2} + \sum_{i=1}^{99} 2\, \underline{Cov(X_i, X_{i+1})} \leq$$

$$100 + 99 \times 2 \times 0.1 \leq 120$$

Equations – Central Limit Theorem

$X_1 \ldots X_N$ independent, identically distributed (i.i.d.) RVs

- Let $S_n = \sum_1^n X_n$, we have $\lim_{n\to\infty} P\left\{\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq c\right\} = \Phi(c)$ for any c
- For examples, refer to Gaussian approximation

Final Review Survey

- Open text

- April 30 (Thur)
    - Final Remarks
    - Review on topics before midterm II

- May 5 (Tue)
    - Review on topics after midterm II

- May 7 (~~Tue~~) Thur.
    - Reading day, no class

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All the rest of this lecture
will not be in the exam

Expectation Maximization

Motivation

- In many cases, we will face a “ chicken & Egg ” problem
- Example
  - Given location of many bees x_i, estimate their bee hive location θ
  - If $x_i \sim N(\theta, \sigma^2)$
  - What if they belong to two hives? ↙
- We need to classify/ separate signals to estimate their parameter
- The estimated parameter can help classify the signals

Application

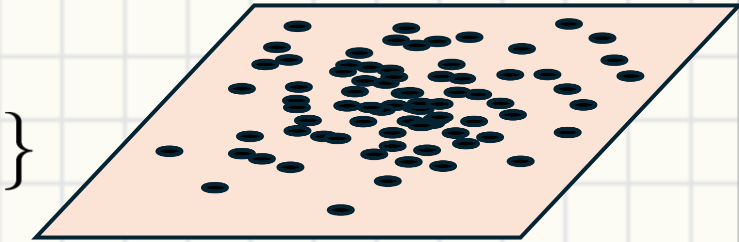
- Unsupervised learning

- Speech recognition
  - Align speech segment to transcriptions

- Image segmentation

- Customer segmentation

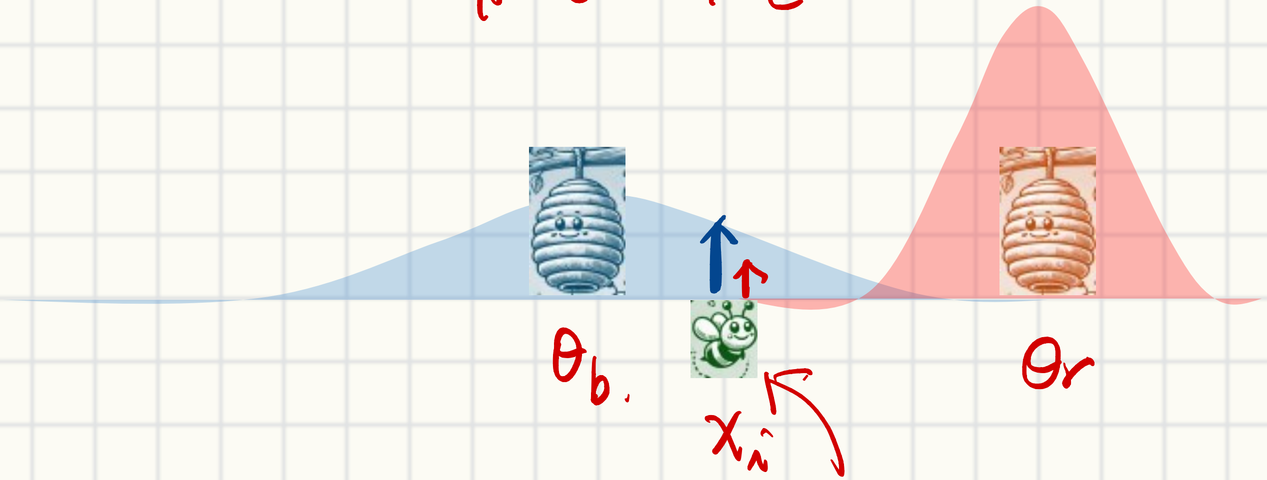
Intuition



Given: Bee locations X Estimate: bee colors Z and bee hives $\{\theta_r, \theta_b\}$

But we can estimate Z if we know locations of both hives

Recall $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{(x-\mu)^2}{\sigma^2}\right\}$



Intuition

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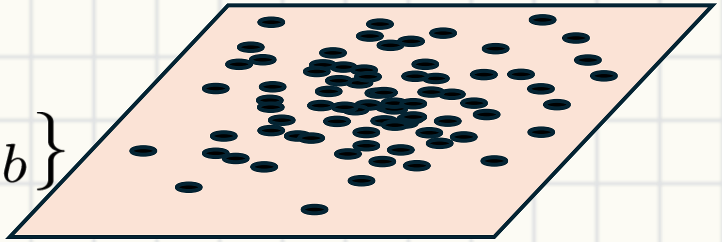
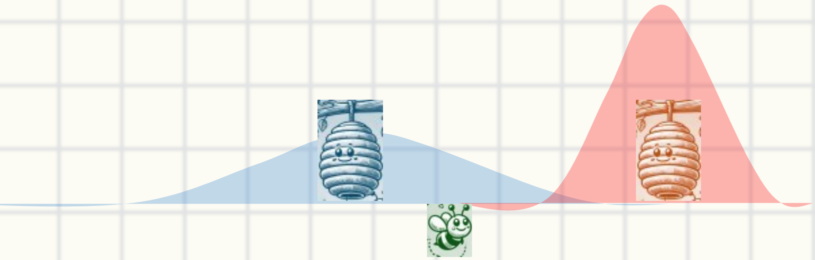
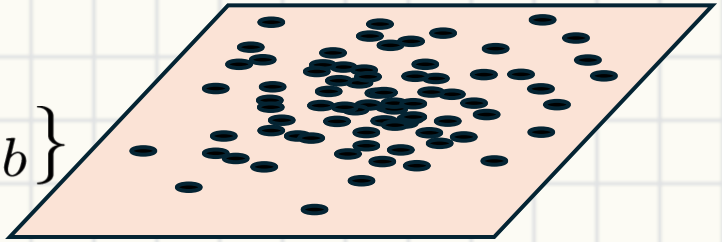


But we can estimate Z **if we know locations of both hives**

$$p(z = r|x, \theta_r, \theta_b) \underset{\text{Bayes}}{=} \frac{p(x|z = r, \theta_r)p(z = r|\theta_r)}{p(x|\theta_r, \theta_b)}$$



Intuition



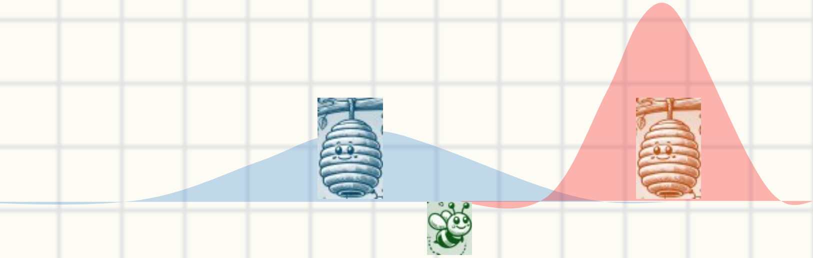
Given: Bee locations X Estimate: bee colors Z and bee hives $\{\theta_r, \theta_b\}$

But we can estimate Z if we know locations of both hives

$$p(z=r|x,\theta_r,\theta_b) = \frac{p(x|z=r,\theta_r)p(z=r|\theta_r)}{p(x|\theta_r,\theta_b)}$$

How likely a red bee is here

How likely a bee is here



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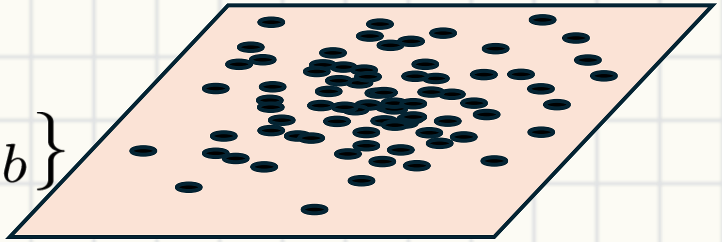

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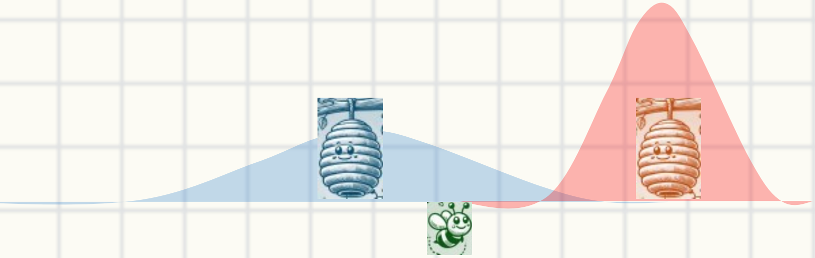
$$p(z=r|x,\theta_r,\theta_b) = \frac{p(x|z=r,\theta_r)p(z=r|\theta_r)}{p(x|\theta_r,\theta_b)}$$

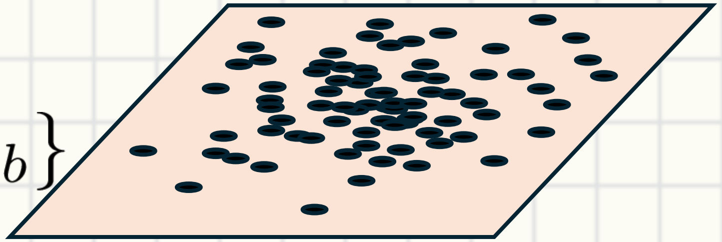
$N(\theta_r, \sigma_r^2)$



$$= \frac{\overbrace{p(x|z=r,\theta_r)}^{\text{Red bee distribution}}\overbrace{p(z=r|\theta_r)}^{\text{Size of the red hive}}}{\underbrace{p(x|z=r,\theta_r)p(z=r|\theta_r)}_{\text{Red bee likelihood}} + \underbrace{p(x|z=b,\theta_b)p(z=b|\theta_b)}_{\text{Blue bee likelihood}}}$$

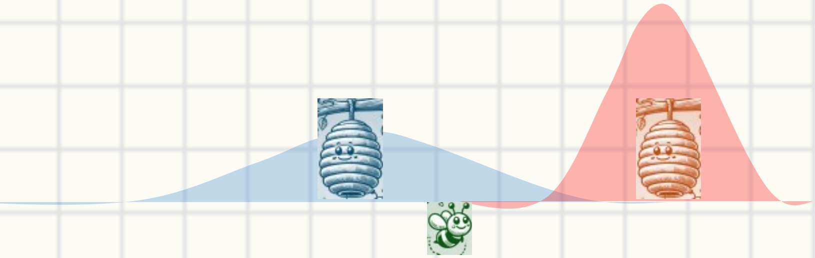
Intuition

Given: Bee locations X **Estimate: bee colors** Z **and bee hives** $\{\theta_r, \theta_b\}$



But we can estimate Z **if we know locations of both hives**

$$p(z = r | x, \theta_r, \theta_b) = \frac{p(x|z = r, \theta_r)p(z = r|\theta_r)}{p(x|\theta_r, \theta_b)}$$



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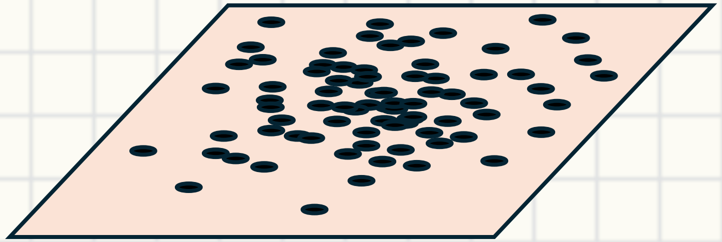
How do we know hive locations? We start with a guess $\theta = \{ \theta_r^{(0)} , \theta_b^{(0)} \}$

That’s our first “chicken”!

Algorithms

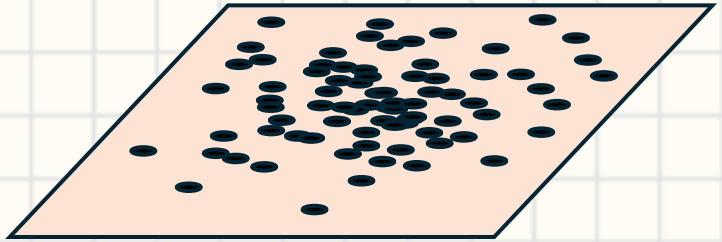


Basic idea:

1. Initialize $\theta = \{ \theta_r^{(0)} , \theta_b^{(0)} \}$ with arbitrary values

2. Write out the likelihood function $p(X, Z|\theta) = p(X|Z, \theta)p(Z|\theta)$ → Somehow need to get Z

Algorithms



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3. Treat Z as a random variable and compute it's distribution $q(z)$

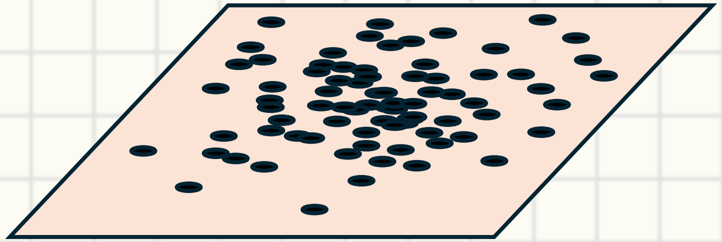
   How? Compute the posterior $q(z) = p(Z \mid X, \theta)$
   (i.e., probability of bin given ball color, probability of bee type given bee location)

4. Treat likelihood as a function of Z. Then, what is average likelihood?

$$\log p(X, Z \mid \theta)q(z) = logp(X|Z, \theta)p(Z|\theta)q(z)$$

$$= \mathbb{E}_{z \sim q(z)}[logp(X|Z, \theta)p(Z|\theta)]$$

Algorithms



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over color.

5. Maximize expected likelihood to get new θ
6. Return to step #1 until convergence

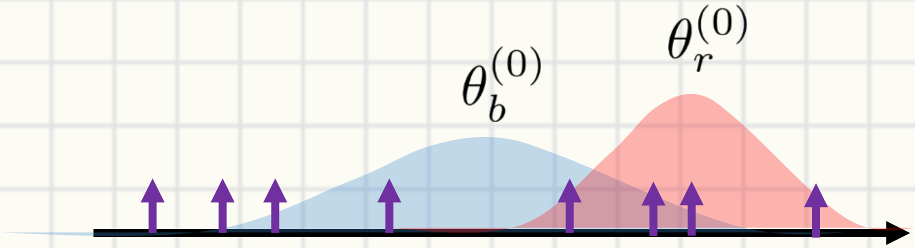
7. Return θ^*

Visualization

E-Step : Given estimated hive locations, Calculate bee color

Given θ, estimate $q(Z) = p(Z|X, \theta)$

$$p(z_n^{(0)} = r | x_n, \theta_r^{(0)}) = \frac{p(x_n | z_n^{(0)} = r, \theta_r^{(0)} p(z^{(0)} = r | \theta_r^{(0)})}{p(x | \theta_r^{(0)}, \theta_b^{(0)})}$$



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prior

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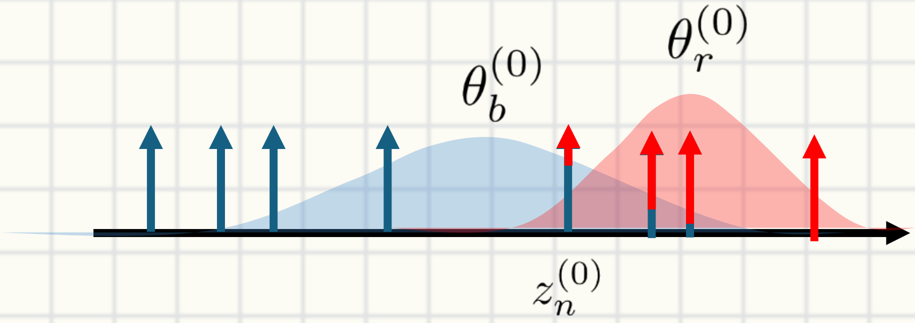

Visualization

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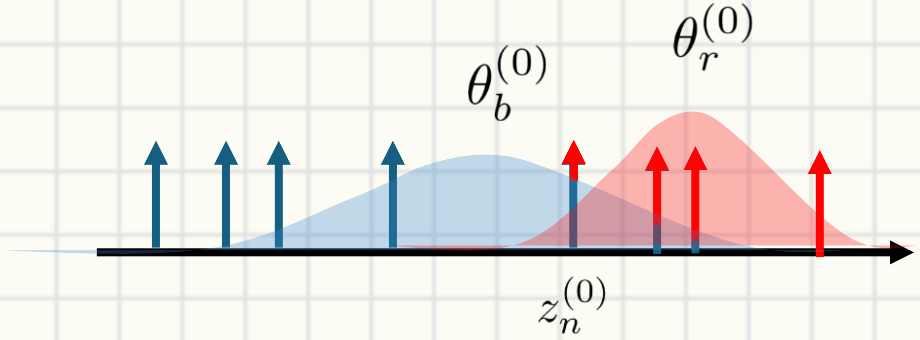
M-Step : Given estimated bee color, Calculate hive locations

Given $q(Z)$, **estimate** θ

$$\theta^{(i+1)} = argmax_\theta \mathbb{E}_{q(Z)}[logp(X|Z, \theta) + logp(Z|\theta)]$$



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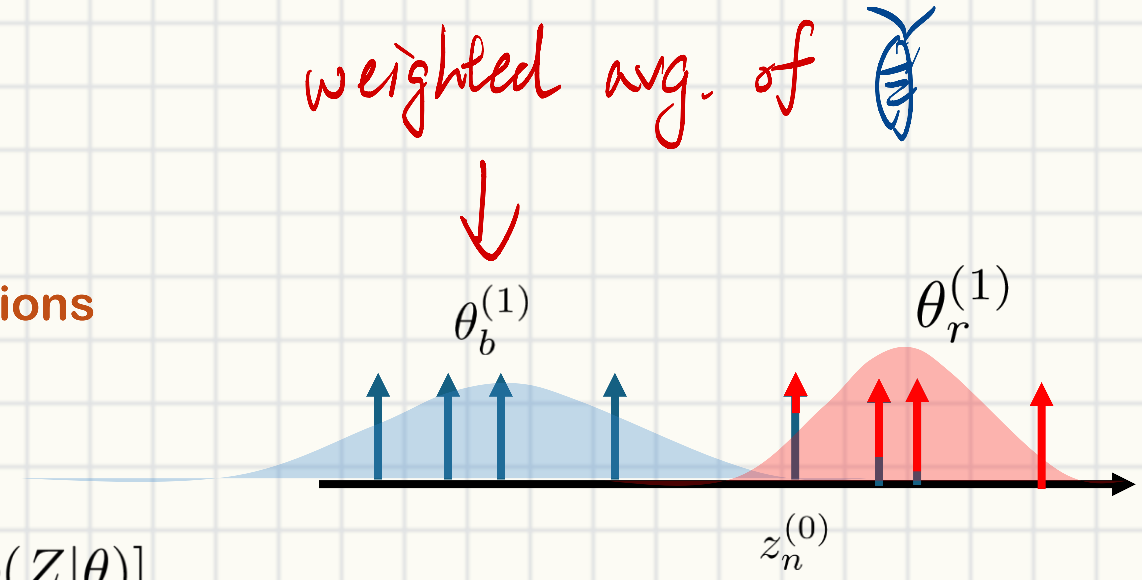

Visualization

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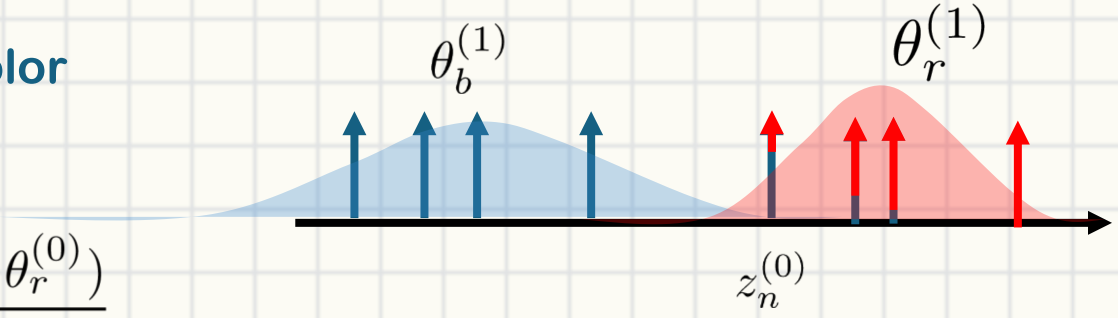

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Visualization

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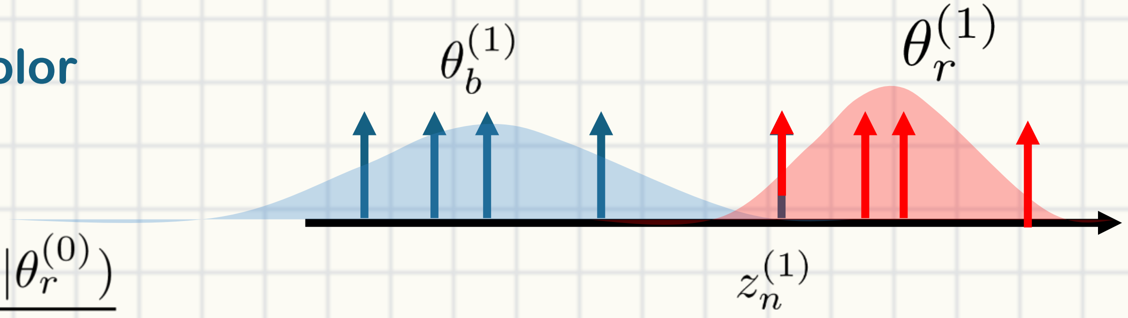

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