Last lecture

Joint PDF (Ch 4.3)

- Definition
- Examples
  - Uniform distribution
  - Conditional distribution

Independent RV (Ch 4.4)

- From event to RV - CDF
- Check using PDF

Agenda

Independent RV (Ch 4.4)

- Product Set
- Examples

Sums of joint RVs (Ch 4.5)

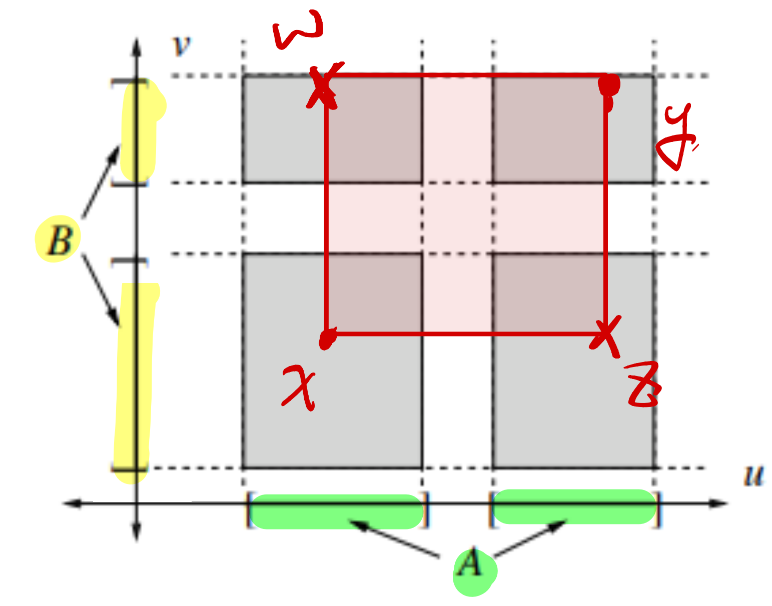
- Motivation
- Examples

More examples on joint RVs (Ch 4.6)

- Max of two RVs
- Buffon’s needle problems

Independent RV

Product Set



Let A, B denote a finite union of intervals

- $|A|$ denotes the total length of $|A|$

The product set $A \times B = \{(u, v): u \in A, v \in B\}$

- The total area $|A \times B| = |A| \times |B|$

product set.

Swap property: $S \in \mathbb{R}^2$ has the swap property if

- For any pair of points $(a, b), (c, d) \in S$, (a, d) and (c, b) also in S

  x y, w z.

Proposition - $S \in \mathbb{R}^2$, S is a product set if and only if it has the swap property

Properties of independent

- If X, Y are independent and jointly continuous type RVs, then support of $f_{X,Y}$ is a product set

support $f_{XY} \triangleq S$

$$f_{XY}(u,v) = f_X(u) \times f_Y(v)$$

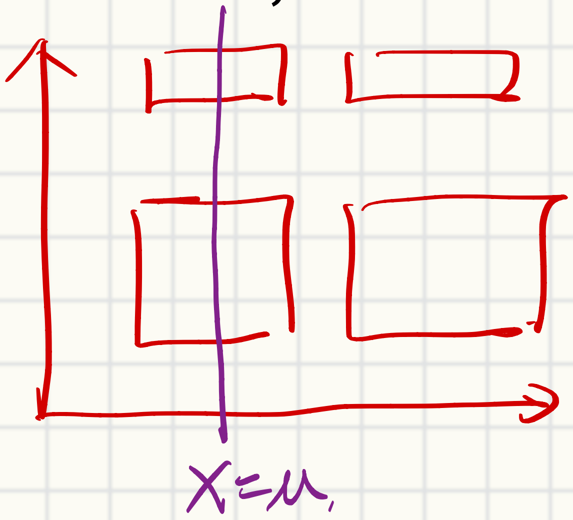
$= A \times B$.

support $\triangleq A$. $\triangleq B$.

- Support X, Y are uniformly distributed over set $S \in \mathbb{R}^2$, then X and Y are independent iif S is a product set

if: $f_{Y|X} = f_Y$.

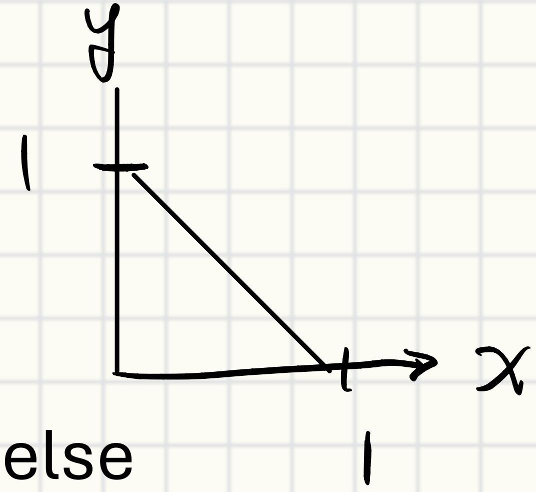
only if:



$f_{Y|X} = \frac{f_{XY}}{f_X}$

Examples

$f_X = \int_0^{1-u} f_{XY}(u,v)\,dv = \int_0^{1-u} Cu^2v^2\,dv$



Decide whether the if X and Y are independent if

- $f_{X,Y}(u,v) = Cu^2v^2$ for $u, v > 0$ and $u + v \leq 1$; 0 else

① $f_{Y|X} = f_Y$? ② S finite set? $= \left[\frac{Cu^2v^3}{3}\right]_0^{1-u} = \frac{Cu^2(1-u)^3}{3}$

$\frac{Cu^2v^2}{\frac{Cu^2(1-u)^3}{3}} \overset{?}{=} \frac{Cv^2(1-v)^3}{3}$

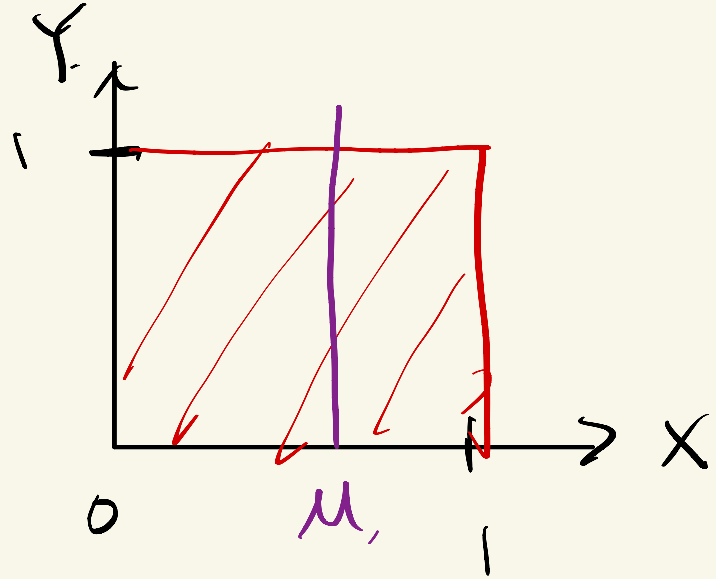
- $f_{X,Y}(u,v) = u + v$ for $u, v \in [0,1]$; 0 else

$\Rightarrow S$ finite? $\Rightarrow f_{Y|X} = f_Y$?

- $f_{X,Y}(u,v) = 9u^2v^2$ for $u, v \in [0,1]$; 0 else

$= f_X(u) \times f_Y(v)$

$\hookrightarrow 3u^2 \qquad \hookrightarrow 3v^2$



$$f_X(u) = \int_0^1 (u+v)\,dv$$

$$= \left[uv + \frac{v^2}{2} \right]_0^1 = u + \frac{1}{2}$$

$$= u + \frac{1}{2}$$

$$f_{Y|X} = \frac{u+v}{u+\frac{1}{2}} \cancel{\overset{?}{\neq}} f_Y(v) = v + \frac{1}{2}$$

$$f_X(u) = \int_0^1 9u^2v^2\,dv = \left[3u^2v^3 \right]_0^1 = 3u^2$$

$$f_Y(v) = 3v^2 \qquad f_{Y|X} = \frac{9u^2v^2}{3u^2} = f_Y(v)$$

Slido

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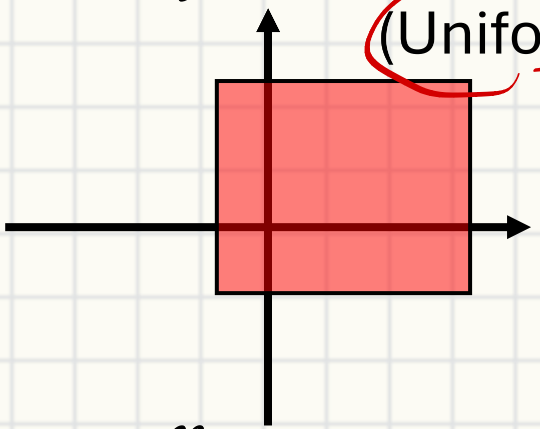
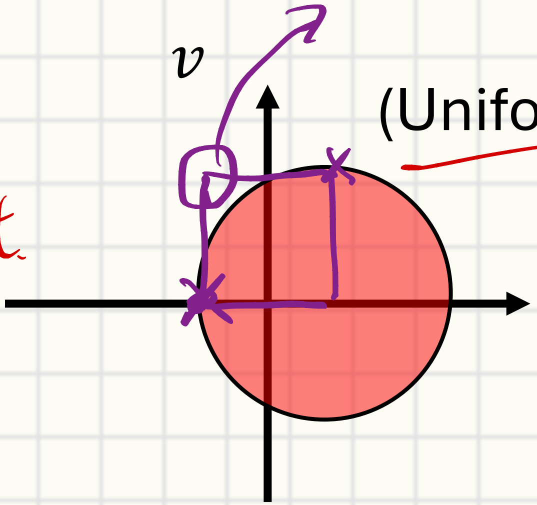
X and Y are independent based if $f_{XY}(u, v)$ is...

(a)



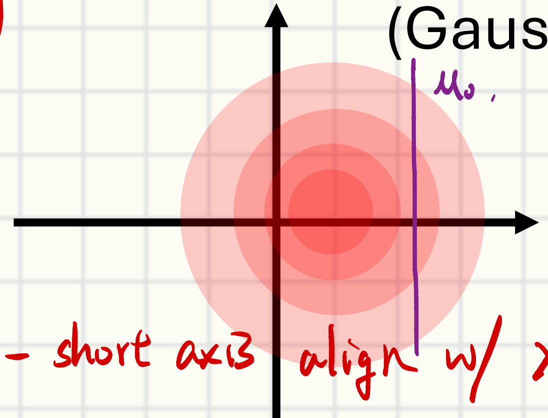
S is product

(b)



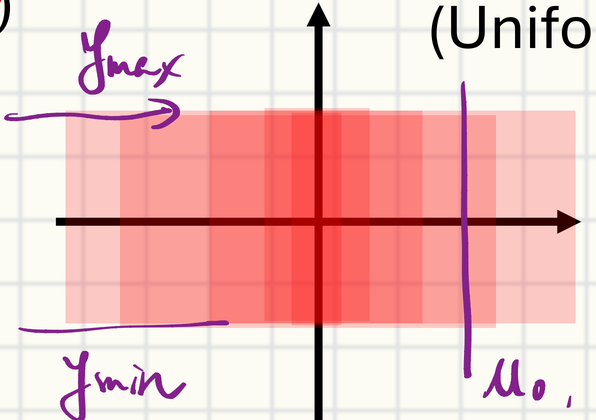
S X product

(c)



Long axis - short axis align w/ X-Y.

(d)



$f_{Y|X} = f_Y$

Sums of joint RVs

Motivation

Recall, we learnt if X and Y are independent

- $E[X+Y] = E[X] + E[Y]$
- $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$

What if we know p_{XY} or f_{XY}? $\Rightarrow$ Independent or not.

- $E[X+Y] = E[X] + E[Y]$ still holds
- What's the sum of your midterm #1 and midterm #2

Sums of Discrete RVs

Let $S = X + Y$

- $p_S(k) = \sum_j P_{XY}(\underbrace{j, k-j}_{X+Y=k})$

  $j \rightarrow X$ outcome

If X and Y are independent, $p_{XY}(j, k-j) = p_X(j)p_Y(k-j)$

- $p_S(k) = \sum_j P_X(j)\, P_Y(k-j) = P_X * P_j$
- Denoted as convolution

Example

Let $X = Bi(n, p)$ and $Y = Bi(m, p)$. $S = X + Y$. Find $p_S(k)$ if X and Y are independent

- Intuitively, $X + Y$ equals "toss a p Head coin $m+n$ times"

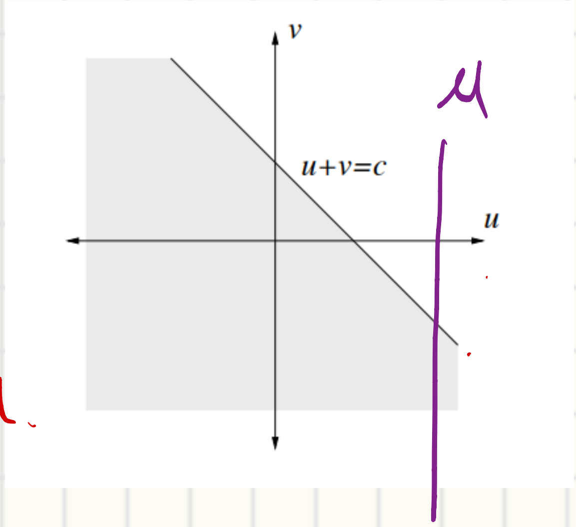
  $S \sim Bi(m+n, p)$

- Verify with formula
- $p_S(k) = \sum_{j=0}^{k} p_X(j) p_Y(k-j)$

$$= \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j} \binom{m}{k-j} p^{k-j} (1-p)^{m-k+j}$$

$$= \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} p^k (1-p)^{m+n-k} = \binom{m+n}{k}$$

Sums of Continuous RVs



Let $S = X + Y$

- $F_S(c) = P\{S \le c\} = \int_{-\infty}^{\infty}\int_{-\infty}^{c-u} f_{XY}\; dv\, du.$
- $f_S(c) = \dfrac{dF_S(c)}{dc} = \int_{-\infty}^{\infty} \Big[f_{XY}\Big]_{0}^{u,\,c-u} du = \int_{-\infty}^{\infty} f_{XY}(u, c-u)\, du$

$\hookrightarrow \dfrac{\int_{-\infty}^{c-u} f_{XY}\, dv}{dc}$

$\downarrow$ Integrate over line $X+Y=c$

If X and Y are independent

- $f_S(c) =$

$$\int_{-\infty}^{\infty} f_X(u) f_Y(c-u)\, du = f_X * f_Y$$

Note: $\dfrac{\int_{-\infty}^{-c-u} f_{XY}\, dv}{dv} = -f_{XY}$

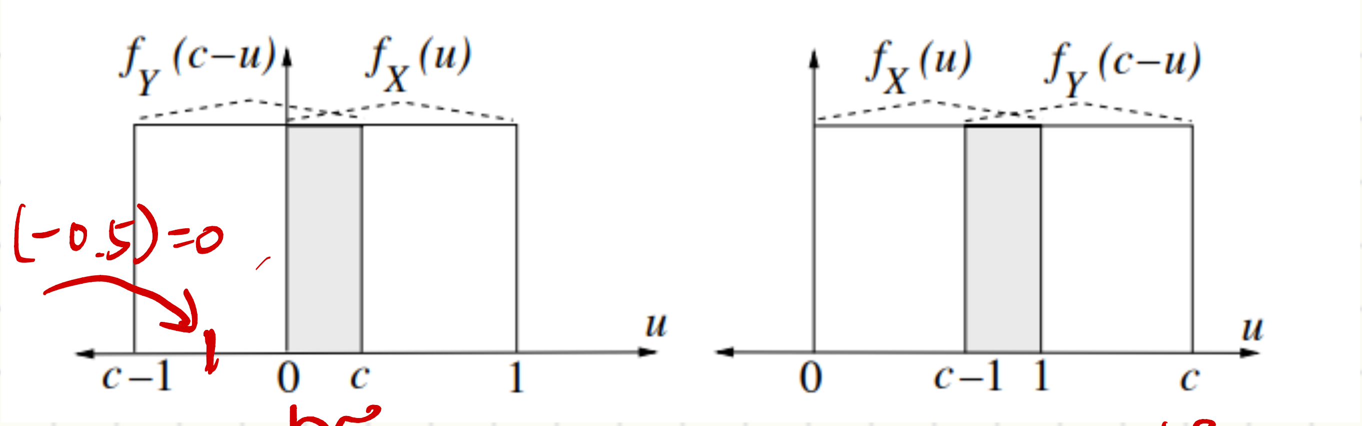
Examples

Suppose X and Y are independent, $X, Y \sim Uniform[0,1]$. Find the pdf of $S = X + Y$ $\in [0,2]$

- $f_S = f_X * f_Y$
- What is $\underline{f_Y(c-u)}$?

$\int_0^1 f_X(u)\, f_Y(c-u)\, du$ $\quad f_Y(-0.5)=0$

$1 \times \|[0,c]\|$



$f_Y > 0$ $\quad 1.8$

- If $0 \leq c \leq 1$, $f_X * f_Y = c$.
- If $1 \leq c \leq 2$, $f_X * f_Y =$

$$f_S(c) = \begin{cases} c & \text{if } c \in [0,1) \\ 2-c & \text{if } c \in [1,2) \\ 0 & \text{else} \end{cases}$$

$\int_0^1 f_X(u)\, f_Y(c-u)\, du = 1 \times \|[c-1,1]\|$

$= 2 - c.$

Notes on Gaussian

Assume $X \sim N(0, \sigma_1^2)$, $Y \sim N(0, \sigma_2^2)$

- Sum of two Gaussian of same mean
  - Mean keeps the same
  - $\sigma_S^2 = \sigma_X^2 + \sigma_Y^2$

$S \sim N(0, \sigma_1^2 + \sigma_2^2)$

  - Tedious proof in textbook formula (4.20)
  - But high-level idea... Approximate by Binomials...

$Var(B_X) = n_X p(1-p) \qquad Var(B_Y) = n_Y p(1-p)$

$Var(B_S) = (n_X + n_Y) p(1-p) = Var(B_X) + Var(B_Y)$