

ECE 313: Exam I

Monday, October 12, 2026

7:00 p.m. — 8:30 p.m.

1. [20 points]

Suppose that a fair coin is independently tossed three times. Define the following events:

 $A = \{\text{The first and third tosses are heads}\},$ $B = \{\text{The second and third tosses are heads}\},$ $C = \{\text{All tosses yield the same outcome}\}.$

- (a) (10 pts) Are A , B , and C pairwise independent? Justify your answer.

Solution: The sample space is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Since the coin is fair and the tosses are independent, all eight outcomes are equally likely. The three events are

$$A = \{HHH, HTH\},$$

$$B = \{HHH, THH\},$$

$$C = \{HHH, TTT\}.$$

Therefore,

$$P(A) = P(B) = P(C) = \frac{2}{8} = \frac{1}{4}.$$

The pairwise intersections are

$$A \cap B = \{HHH\},$$

$$A \cap C = \{HHH\},$$

$$B \cap C = \{HHH\}.$$

Hence,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{8}.$$

However,

$$P(A)P(B) = P(A)P(C) = P(B)P(C) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

Since

$$\frac{1}{8} \neq \frac{1}{16},$$

none of the three pairs is independent.

Therefore,

A , B , and C are not pairwise independent.

(b) (10 pts) Are A , B , and C mutually independent? Justify your answer.

Solution: For three events to be mutually independent, they must be pairwise independent and satisfy

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

From part (a), the three events are not pairwise independent. Therefore, they cannot be mutually independent.

We can also verify this directly.

Since

$$A \cap B \cap C = \{HHH\},$$

we have

$$P(A \cap B \cap C) = \frac{1}{8}.$$

On the other hand,

$$P(A)P(B)P(C) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}.$$

Thus,

$$P(A \cap B \cap C) = \frac{1}{8} \neq \frac{1}{64} = P(A)P(B)P(C).$$

Therefore,

A , B , and C are not mutually independent.

2. [25 points]

Suppose the random variable X has the pmf

$$p_X(k) = \begin{cases} c(4-k), & k = 1, 2, 3, \\ d, & k = m, \\ 0, & \text{otherwise,} \end{cases}$$

where $m \notin \{1, 2, 3\}$ is an integer.

- (a) (5 pts) Given that $p_X(m) = 2p_X(2)$, find the constants c and d so that p_X is a valid pmf.

Solution: The values on $\{1, 2, 3\}$ are

$$p_X(1) = 3c, \quad p_X(2) = 2c, \quad p_X(3) = c.$$

The condition gives $d = p_X(m) = 2p_X(2) = 4c$. The total probability must be 1:

$$3c + 2c + c + 4c = 10c = 1 \implies c = \frac{1}{10}, \quad d = \frac{4}{10} = \frac{2}{5}.$$

So $p_X(1) = 0.3$, $p_X(2) = 0.2$, $p_X(3) = 0.1$, $p_X(m) = 0.4$.

- (b) (5 pts) Find m such that $\mathbb{E}[X] = -1$.

Solution:

$$\mathbb{E}[X] = 1(0.3) + 2(0.2) + 3(0.1) + m(0.4) = 0.3 + 0.4 + 0.3 + 0.4m = 1 + 0.4m.$$

Setting $1 + 0.4m = -1$ gives $0.4m = -2$, so

$$m = -5.$$

(Note $m = -5 \notin \{1, 2, 3\}$, as required.)

- (c) (5 pts) Using the values from (a) and (b), find $\text{Var}(X)$.

Solution:

$$\mathbb{E}[X^2] = 1^2(0.3) + 2^2(0.2) + 3^2(0.1) + (-5)^2(0.4) = 0.3 + 0.8 + 0.9 + 10 = 12.$$

Therefore

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 12 - (-1)^2 = 11.$$

- (d) (10 pts) Let $Z = (X - 2)^2$. Find the pmf of Z and compute $\mathbb{E}[Z]$ in two ways: directly from the pmf of Z , and using only $\mathbb{E}[X]$ and $\text{Var}(X)$.

Solution: Map each value of X :

$$X = 1 \mapsto Z = 1, \quad X = 2 \mapsto Z = 0, \quad X = 3 \mapsto Z = 1, \quad X = -5 \mapsto Z = 49.$$

Combining values of X that map to the same Z ,

$$p_Z(0) = 0.2, \quad p_Z(1) = 0.3 + 0.1 = 0.4, \quad p_Z(49) = 0.4, \quad p_Z(z) = 0 \text{ otherwise.}$$

Directly: $\mathbb{E}[Z] = 0(0.2) + 1(0.4) + 49(0.4) = 0.4 + 19.6 = 20.$

Using moments: $\mathbb{E}[(X - 2)^2] = \mathbb{E}[X^2] - 4\mathbb{E}[X] + 4 = 12 + 4 + 4 = 20.$ Equivalently,

$$\mathbb{E}[(X - 2)^2] = \text{Var}(X) + (\mathbb{E}[X] - 2)^2 = 11 + (-3)^2 = 20.$$

Both methods agree: $\mathbb{E}[Z] = 20.$

3. [25 points]

The number of requests X arriving at a web server in one second is a nonnegative integer-valued random variable with $\mathbb{E}[X] = 20$ and $\text{Var}(X) = 16$.

- (a) (6 pts) Using only the mean, upper bound $P(X \geq 30)$.

Solution: By Markov's inequality with $a = 30$,

$$P(X \geq 30) \leq \frac{\mathbb{E}[X]}{30} = \frac{20}{30} = \frac{2}{3} \approx 0.667.$$

- (b) (6 pts) Using the mean and variance, upper bound $P(X \geq 30)$. Which bound is tighter?

Solution: If $X \geq 30$, then $X - 20 \geq 10$, so $|X - 20| \geq 10$. By Chebyshev's inequality,

$$P(X \geq 30) \leq P(|X - 20| \geq 10) \leq \frac{\text{Var}(X)}{10^2} = \frac{16}{100} = 0.16.$$

Chebyshev's bound is much tighter, since it uses the variance, i.e., the information that X is concentrated around its mean.

- (c) (6 pts) Find a lower bound on $P(12 < X < 28)$.

Solution:

$$P(12 < X < 28) = P(|X - 20| < 8) = 1 - P(|X - 20| \geq 8) \geq 1 - \frac{16}{8^2} = 1 - \frac{1}{4} = \frac{3}{4}.$$

- (d) (7 pts) Let $\bar{X}_n$ be the average (sample mean) of the request counts over n independent seconds, each distributed like X . How large must n be to guarantee, by Chebyshev's inequality, that $P(|\bar{X}_n - 20| \geq 1) \leq 0.01$?

Solution: $\mathbb{E}[\bar{X}_n] = 20$ and, by independence, $\text{Var}(\bar{X}_n) = \frac{1}{n^2} \cdot n \cdot 16 = \frac{16}{n}$. By Chebyshev,

$$P(|\bar{X}_n - 20| \geq 1) \leq \frac{16/n}{1^2} = \frac{16}{n}.$$

Requiring $\frac{16}{n} \leq 0.01$ gives

$$n \geq 1600.$$

4. [30 points]

A coin is flipped 4 times and the number of heads X is recorded. Under hypothesis H_0 the coin is fair, so $X \sim \text{Binomial}(4, \frac{1}{2})$. Under H_1 the coin is biased, so $X \sim \text{Binomial}(4, \frac{3}{4})$.

- (a) (10 pts) Complete the likelihood matrix (the table of $P(X = k | H_i)$ for $k = 0, \dots, 4$), compute the likelihood ratio $\Lambda(k) = P(X = k | H_1)/P(X = k | H_0)$, and show that any likelihood ratio test has the form “decide H_1 if $X \geq k^*$.”

Solution: Under H_0 : $P(X = k | H_0) = \binom{4}{k} \frac{1}{16}$. Under H_1 : $P(X = k | H_1) = \binom{4}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{4-k} = \binom{4}{k} \frac{3^k}{256}$.

k	0	1	2	3	4
H_0	$\frac{16}{256}$	$\frac{64}{256}$	$\frac{96}{256}$	$\frac{64}{256}$	$\frac{16}{256}$
H_1	$\frac{1}{256}$	$\frac{12}{256}$	$\frac{54}{256}$	$\frac{108}{256}$	$\frac{81}{256}$

The binomial coefficients cancel in the ratio:

$$\Lambda(k) = \frac{\binom{4}{k} 3^k / 256}{\binom{4}{k} / 16} = \frac{3^k}{16}.$$

Since $3 > 1$, $\Lambda(k)$ is strictly increasing in k , so $\{\Lambda(k) \geq \eta\}$ is always a set of the form $\{k \geq k^*\}$ for some integer k^* .

- (b) (10 pts) Find the ML decision rule and compute $p_{\text{false alarm}}$ and p_{miss} .

Solution: ML decides H_1 when $\Lambda(k) \geq 1$, i.e., $3^k \geq 16$, i.e.,

$$k \geq \frac{\ln 16}{\ln 3} \approx 2.52.$$

So the ML rule is: decide H_1 if $X \geq 3$. (Check from the table: in columns 3, 4 the H_1 entry is larger; in columns 0, 1, 2 the H_0 entry is larger.)

$$p_{\text{false alarm}} = P(X \geq 3 | H_0) = \frac{64 + 16}{256} = \frac{5}{16} = 0.3125,$$

$$p_{\text{miss}} = P(X \leq 2 | H_1) = \frac{1 + 12 + 54}{256} = \frac{67}{256} \approx 0.262.$$

- (c) (10 pts) Suppose the priors are $\pi_0 = P(H_0) = 0.8$ and $\pi_1 = P(H_1) = 0.2$. Find the MAP rule and its average error probability p_e . Compare with p_e for the ML rule under the same priors.

Solution: MAP decides H_1 when $\pi_1 P(X = k | H_1) \geq \pi_0 P(X = k | H_0)$, i.e.,

$$\Lambda(k) \geq \frac{\pi_0}{\pi_1} = 4 \iff 3^k \geq 64 \iff k \geq \frac{\ln 64}{\ln 3} \approx 3.79.$$

So the MAP rule is: decide H_1 if $X = 4$. For this rule

$$p_{\text{false alarm}} = P(X = 4 | H_0) = \frac{1}{16} = 0.0625, \quad p_{\text{miss}} = P(X \leq 3 | H_1) = 1 - \frac{81}{256} = \frac{175}{256} \approx 0.684.$$

$$p_e^{\text{MAP}} = \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss}} = 0.8(0.0625) + 0.2 \left(\frac{175}{256} \right) \approx 0.050 + 0.137 = 0.187.$$

For the ML rule,

$$p_e^{\text{ML}} = 0.8 \left(\frac{5}{16} \right) + 0.2 \left(\frac{67}{256} \right) \approx 0.250 + 0.052 = 0.302.$$

The MAP rule has a substantially smaller error probability, as it must, since MAP minimizes p_e for the given priors. (Sanity check: the trivial rule “always decide H_0 ” has $p_e = \pi_1 = 0.2$, which is also worse than MAP.)