

ECE 313: Problem Set 7

Due: Friday, Oct. 30 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes*, Sections 3.5 - 3.8

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME

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SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Poisson process probabilities]**

Consider a Poisson process with arrival rate $\lambda > 0$.

- (a) Find the probability there is exactly one arrival in each of the intervals $(0, 1]$, $(1, 2]$, and $(2, 3]$.
- (b) Find the probability that there are two arrivals in the interval $(0, 2]$ and two arrivals in the interval $(1, 3]$. (*Hint:* The occurrence of the event in (a) implies the occurrence of the event in (b). What other ways can the event in (b) occur?)
- (c) Find the probability that there are two arrivals in the interval $(1, 2]$, given that there are two arrivals in the interval $(0, 2]$ and two arrivals in the interval $(1, 3]$.

2. **[Central limit theorem and Gaussian approximation]**

- (a) A wafer contains 400 chips. Each is defective with probability 0.1, independently. Let X be the number of defective chips. Use the Gaussian approximation with continuity correction to estimate $P\{X \leq 45\}$ and $P\{35 \leq X \leq 45\}$.
- (b) An accountant adds 100 numbers, each first rounded to the nearest integer. The rounding errors are independent and uniformly distributed over $[-0.5, 0.5]$. Use the Gaussian approximation to estimate the probability that the total rounding error exceeds 5 in magnitude.
- (c) Packets arrive at a router according to a Poisson process with rate $\lambda = 3$ packets per millisecond (ms). Let T_{100} be the arrival time of the 100th packet. Find $E[T_{100}]$ and $\text{Var}(T_{100})$, and use the Gaussian approximation to estimate $P\{T_{100} > 40\}$.

- (d) For part (c), write the exact probability $P\{T_{100} > 40\}$ as a Poisson sum. The exact value is 0.0279. Briefly explain why the Gaussian approximation is less accurate here than in part (a).
3. **[ML estimation for a Gaussian distribution]**
 The random variable X is assumed to have a Gaussian distribution. It is observed that $X = \sqrt{2}$.
- (a) Suppose the variance of X is assumed to be one. Find the ML estimate of its mean given the observation.
- (b) (For this part we assume the mean is known and the variance is to be estimated, for the same observation $X = \sqrt{2}$.) Suppose the mean of X is assumed to be zero. Find the ML estimate of the variance of X given the observation.
4. **[Transformation of a uniform distribution]**
 X denotes a uniform random variable with mean 1 and variance 3. Find the pdf of $Y = |X|$.
5. **[A binary quantizer with Laplacian input]**
 Suppose X has pdf $f_X(u) = \frac{e^{-|u|}}{2}$ and $Y = g(X)$ where $g(u) = \alpha(\text{sign}(u)) = \begin{cases} \alpha & \text{if } u \geq 0 \\ -\alpha & \text{if } u < 0 \end{cases}$ for some constant α . So Y is the output of a binary quantizer with input X .
- (a) Describe the pdf or pmf of Y .
- (b) Find the mean square quantization error, $E[(X - Y)^2]$. Your answer should depend on α . (Hint: $\int_0^\infty u^k e^{-u} du = k!$ for nonnegative integers k .)
- (c) Find α to minimize the mean square quantization error.
6. **[Function of a random variable]**
 Let X have pdf $f_X(u) = \frac{1}{2u^2}$ for $|u| \geq 1$ and $f_X(u) = 0$ for $|u| < 1$. Let $Y = \sqrt{|X|}$.
- (a) Using LOTUS, find $E[Y]$.
- (b) Find the pdf of Y .
- (c) (This part does not involve Y .) Find the nondecreasing function h so that $h(X)$ is uniformly distributed over $[0, 1]$. Be as explicit as possible.