University of Illinois **Fall 2026**

ECE 313: Problem Set 6: Problems and Solutions

Due: Sunday, Oct 25 at 06:59:00 p.m.
Reading: *ECE 313 Course Notes, Sections 3.3-3.6.2*

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Uniform distribution]**
Let X be uniformly distributed over an interval $[a, b]$ with $E[X] = 4$ and $\mathrm{Var}(X) = 3$.

   (a) Find a and b, and write the pdf f_X.
   Solution: For X uniform on $[a, b]$, $E[X] = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$. So $a + b = 8$ and $(b - a)^2 = 36$, giving $b - a = 6$. Solving, $a = 1$ and $b = 7$:

$$f_X(u) = \begin{cases} \frac{1}{6}, & 1 \le u \le 7, \\ 0, & \text{else.} \end{cases}$$

   (b) Find $P\{X > 5\}$.
   Solution: $P\{X > 5\} = \frac{7-5}{6} = \frac{1}{3}$.

   (c) Find $P(X > 5 \mid X > 3)$. Is the conditional distribution of X given $X > 3$ still uniform? Over what interval?
   Solution: Since $\{X > 5\} \subset \{X > 3\}$,

$$P(X > 5 \mid X > 3) = \frac{P\{X > 5\}}{P\{X > 3\}} = \frac{2/6}{4/6} = \frac{1}{2}.$$

   For $3 \le c \le 7$, $P(X \le c \mid X > 3) = \frac{(c-3)/6}{4/6} = \frac{c-3}{4}$, which is the CDF of the uniform distribution over $[3, 7]$. Conditioning a uniform random variable on a subinterval leaves it uniform over that subinterval. Unlike the exponential distribution, the uniform distribution is not memoryless.

(d) Find the value c such that $P\{X \le c\} = 0.75$.

Solution: $F_X(c) = \frac{c-1}{6} = 0.75$ gives $c = 5.5$.

(e) A stick of unit length is broken at a point U, uniformly distributed over $[0, 1]$. Find the probability that the longer piece is at least twice as long as the shorter piece. Also find the expected length of the longer piece.

Solution: The pieces have lengths U and $1-U$. Since the lengths sum to one, the longer piece is at least twice the shorter exactly when the shorter piece is at most $1/3$:

$$P\{\min(U, 1-U) \le 1/3\} = P\{U \le 1/3\} + P\{U \ge 2/3\} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}.$$

By LOTUS and symmetry about $1/2$,

$$E[\max(U, 1-U)] = \int_0^{1/2} (1-u)\,du + \int_{1/2}^1 u\,du = 2\int_{1/2}^1 u\,du = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

Alternatively, the longer piece is uniformly distributed over $[1/2, 1]$, so its mean is the midpoint $3/4$.

2. **[Exponential distribution, memoryless property]**
The lifetime T (in hours) of a laser diode is exponentially distributed with parameter λ. Its median lifetime is 2000 hours, meaning $P\{T \le 2000\} = 1/2$.

(a) Find the parameter λ.

Solution: For $t \ge 0$, $P\{T > t\} = e^{-\lambda t}$. The median condition gives $e^{-2000\lambda} = 1/2$, so

$$\lambda = \frac{\ln 2}{2000} \approx 3.466 \times 10^{-4} \text{ per hour.}$$

As in Example 3.4.1, a convenient form is $P\{T > t\} = e^{-(\ln 2)t/2000} = 2^{-t/2000}$ for $t \ge 0$.

(b) Find the mean and standard deviation of T.

Solution: For the exponential distribution the mean and standard deviation are both $1/\lambda$:

$$E[T] = \sigma_T = \frac{2000}{\ln 2} \approx 2885 \text{ hours.}$$

The mean exceeds the median because the exponential pdf has a long right tail.

(c) Find $P\{T > 3000\}$. Express it in the simplest form.

Solution: $P\{T > 3000\} = 2^{-3000/2000} = 2^{-1.5} = \frac{1}{2\sqrt{2}} \approx 0.3536$.

(d) A diode has already operated for 1500 hours without failing. Find the probability it is still working at hour 4000.

Solution: By the memoryless property, $P(T > s + t \mid T > s) = P\{T > t\}$. With $s = 1500$ and $t = 2500$,

$$P(T > 4000 \mid T > 1500) = \frac{P\{T > 4000\}}{P\{T > 1500\}} = \frac{2^{-2}}{2^{-0.75}} = 2^{-1.25} = P\{T > 2500\} \approx 0.4204.$$

The 1500 hours already survived do not "wear out" the diode.

(e) The diode is inspected only at the end of each hour, so failure is detected at hour $K = \lceil T \rceil$ (T rounded up to an integer). Show that K is geometrically distributed and find its parameter p.

Solution: For each integer $k \geq 1$, $\{K = k\} = \{k - 1 < T \leq k\}$. So

$$P\{K = k\} = e^{-\lambda(k-1)} - e^{-\lambda k} = \left(e^{-\lambda}\right)^{k-1}\left(1 - e^{-\lambda}\right), \qquad k = 1, 2, \ldots$$

This is the geometric pmf with parameter $p = 1 - e^{-\lambda} = 1 - 2^{-1/2000} \approx 3.465 \times 10^{-4}$. Because λ is small, $p \approx \lambda$, matching the connection in Section 3.4 between scaled geometric and exponential distributions. As a check, $E[K] = 1/p \approx 2885.9$ hours, about half an hour more than $E[T]$, as expected from rounding up.

3. **[Poisson processes]**
Packets arrive at a router according to a Poisson process with rate $\lambda = 3$ packets per millisecond (ms). Let N_t be the number of arrivals in $[0, t]$, with t in ms.

(a) Find the probability that no packets arrive in the first 0.5 ms.

Solution: $N_{0.5}$ has the $Poi(1.5)$ distribution, so $P\{N_{0.5} = 0\} = e^{-1.5} \approx 0.2231$. Equivalently, this is $P\{U_1 > 0.5\}$ for the exponential first intercount time.

(b) Find the probability that exactly 4 packets arrive in $(0, 1]$ and exactly 2 packets arrive in $(1, 1.5]$.

Solution: The counts in $(0, 1]$ and $(1, 1.5]$ are independent with $Poi(3)$ and $Poi(1.5)$ distributions:

$$\frac{e^{-3}3^4}{4!} \cdot \frac{e^{-1.5}(1.5)^2}{2!} = (3.375\, e^{-3})(1.125\, e^{-1.5}) = 3.797\, e^{-4.5} \approx 0.1680 \times 0.2510 \approx 0.0422.$$

(c) Find $P(N_2 = 6 \mid N_1 = 4)$.

Solution: Given $N_1 = 4$, the event $N_2 = 6$ means exactly 2 arrivals in $(1, 2]$. By independent increments,

$$P(N_2 = 6 \mid N_1 = 4) = P\{N_2 - N_1 = 2\} = \frac{e^{-3}3^2}{2!} = 4.5\, e^{-3} \approx 0.2240.$$

(d) Find $P(N_1 = 2 \mid N_2 = 6)$. Identify the distribution family and its parameters.

Solution: Write $\{N_1 = 2, N_2 = 6\} = \{N_1 = 2,\ N_2 - N_1 = 4\}$ and use independence with $N_2 \sim Poi(6)$:

$$\begin{aligned} P(N_1 = 2 \mid N_2 = 6) &= \frac{\frac{e^{-3}3^2}{2!} \cdot \frac{e^{-3}3^4}{4!}}{\frac{e^{-6}6^6}{6!}} \\ &= \frac{6!}{2!\,4!} \cdot \frac{3^6}{6^6} = \binom{6}{2}\left(\frac{1}{2}\right)^6 = \frac{15}{64} \approx 0.2344. \end{aligned}$$

This is the binomial pmf with $n = 6$ and $p = 1/2$ evaluated at 2. As in Example 3.5.3, given 6 counts in $[0, 2]$, each count independently falls in $[0, 1]$ with probability $1/2$. Note that λ cancels out.

(e) Given that exactly one packet arrived during $[0, 1]$, find the conditional probability that it arrived during $[0, 0.25]$.

Solution: Split $[0, 1]$ into $[0, 0.25]$ and $(0.25, 1]$:

$$P(N_{0.25} = 1 \mid N_1 = 1) = \frac{P\{N_{0.25} = 1\}\, P\{N_1 - N_{0.25} = 0\}}{P\{N_1 = 1\}} = \frac{(0.75\, e^{-0.75})(e^{-2.25})}{3\, e^{-3}} = 0.25.$$

The answer equals the fraction of the interval. Given one count in $[0, 1]$, its arrival time is uniformly distributed over $[0, 1]$.

4. **[Erlang distribution]**

A satellite subsystem uses one active component plus two cold spares. When the active component fails, a spare is switched in instantly (spares do not fail while unused). Component lifetimes are independent and exponentially distributed with mean 100 hours. Let T be the time at which the third (last) component fails.

(a) Identify the distribution of T and write its pdf.

Solution: $T = U_1 + U_2 + U_3$, a sum of three independent exponential lifetimes with $\lambda = 1/100 = 0.01$ per hour. Equivalently, T is the time of the third count of a Poisson process with rate 0.01. So T has the Erlang distribution with $r = 3$ and $\lambda = 0.01$:

$$\begin{aligned} f_T(t) &= \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!} = \frac{(0.01)^3\, t^2\, e^{-0.01t}}{2} \\ &= 5 \times 10^{-7}\, t^2 e^{-0.01t}, \qquad t \geq 0, \end{aligned}$$

and $f_T(t) = 0$ for $t < 0$.

(b) Find $E[T]$, $\mathrm{Var}(T)$, and the standard deviation of T.

Solution: $E[T] = r/\lambda = 300$ hours, $\mathrm{Var}(T) = r/\lambda^2 = 30{,}000$ hours^2, and $\sigma_T = \sqrt{30{,}000} \approx 173.2$ hours. Both follow from adding the means and variances of three independent exponential random variables.

(c) Find $P\{T > 200\}$.

Solution: Use the identity from Section 3.5.3: $\{T_r > t\} = \{N_t \leq r - 1\}$. Here $N_{200} \sim Poi(\lambda t) = Poi(2)$, so

$$P\{T > 200\} = P\{N_{200} \leq 2\} = e^{-2}\left(1 + 2 + \frac{2^2}{2!}\right) = 5e^{-2} \approx 0.6767.$$

(d) Find $P\{T \leq 500\}$.

Solution: Now $N_{500} \sim Poi(5)$:

$$P\{T \leq 500\} = 1 - P\{N_{500} \leq 2\} = 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2!}\right) = 1 - 18.5\, e^{-5} \approx 0.8753.$$

(e) Find the mode of T (the value of t that maximizes $f_T(t)$).

Solution: Maximize $g(t) = t^2 e^{-0.01t}$. Setting $g'(t) = (2t - 0.01t^2)e^{-0.01t} = 0$ gives $t = 200$ hours. Since $f_T(0) = 0$ and $f_T(t) \to 0$ as $t \to \infty$, this is the maximum. In general the mode of the Erlang(r, λ) distribution is $(r - 1)/\lambda$, which is less than the mean r/λ.

(f) Suppose the designer may use r components in total (one active plus $r-1$ spares). Find the smallest r such that $P\{T_r > 200\} \geq 0.95$, where T_r is the time at which the r-th component fails.

Solution: $P\{T_r > 200\} = P\{N_{200} \leq r-1\}$ with $N_{200} \sim Poi(2)$. The $Poi(2)$ CDF is

k	0	1	2	3	4	5
$P\{N_{200} \leq k\}$	0.1353	0.4060	0.6767	0.8571	0.9473	0.9834

The first value at or above 0.95 occurs at $k = r-1 = 5$. So $r = 6$ components (one active plus five spares) are needed; five components give only 0.9473.

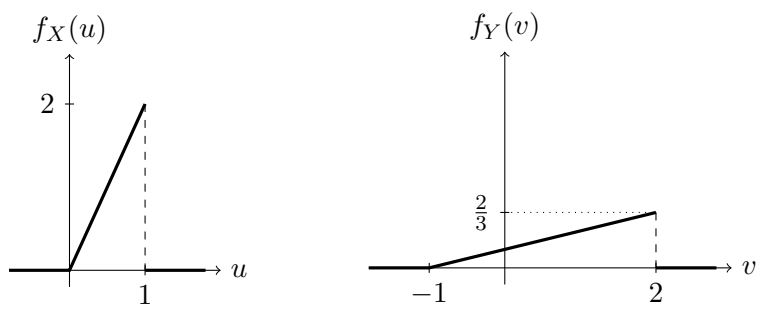
5. **[Scaling rule for pdfs]**
Let X have pdf $f_X(u) = 2u$ for $0 \leq u \leq 1$ and $f_X(u) = 0$ otherwise.

(a) Let $Y = 3X - 1$. Use the scaling rule to find f_Y. Sketch f_X and f_Y, and check that f_Y integrates to one.

Solution: With $a = 3$ and $b = -1$, $X \in [0,1]$ maps to $Y \in [-1,2]$, and

$$f_Y(v) = \frac{1}{3} f_X\left(\frac{v+1}{3}\right) = \begin{cases} \frac{2(v+1)}{9}, & -1 \leq v \leq 2, \\ 0, & \text{else.} \end{cases}$$

Sketch: f_X is a ramp rising from 0 at $u = 0$ to height 2 at $u = 1$. f_Y is a ramp rising from 0 at $v = -1$ to height $2/3$ at $v = 2$: it is f_X stretched horizontally by 3, shrunk vertically by 3, then shifted left by 1.



Check: $\int_{-1}^{2} \frac{2(v+1)}{9}\, dv = \frac{2}{9} \cdot \frac{3^2}{2} = 1$.

(b) Find $E[Y]$ and $\mathrm{Var}(Y)$ two ways: from f_Y directly, and from $E[X]$, $\mathrm{Var}(X)$, and the linear scaling relations.

Solution: Directly, substituting $w = v + 1 \in [0,3]$:

$$E[Y] = \int_0^3 (w-1)\frac{2w}{9}\, dw = \frac{2}{9}\left(9 - \frac{9}{2}\right) = 1,$$

$$E[Y^2] = \int_0^3 (w-1)^2 \frac{2w}{9}\, dw = \frac{2}{9}\left(\frac{81}{4} - 18 + \frac{9}{2}\right) = \frac{3}{2},$$

so $\mathrm{Var}(Y) = \frac{3}{2} - 1 = \frac{1}{2}$. Via scaling: $E[X] = \int_0^1 2u^2\, du = \frac{2}{3}$ and $E[X^2] = \int_0^1 2u^3\, du = \frac{1}{2}$, so $\mathrm{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$. Then $E[Y] = 3 \cdot \frac{2}{3} - 1 = 1$ and $\mathrm{Var}(Y) = 3^2 \cdot \frac{1}{18} = \frac{1}{2}$. The two methods agree.

(c) Let $W = -2X + 1$. The scaling rule (3.5) requires $a > 0$, so derive f_W directly from the CDF.

Solution: Since $-2 < 0$, the inequality flips when solving for X:

$$F_W(c) = P\{-2X + 1 \le c\} = P\left\{X \ge \frac{1-c}{2}\right\} = 1 - F_X\left(\frac{1-c}{2}\right).$$

Differentiating with the chain rule gives $f_W(c) = \frac{1}{2} f_X\left(\frac{1-c}{2}\right)$. Since $X \in [0,1]$ maps to $W \in [-1,1]$,

$$f_W(c) = \begin{cases} \frac{1-c}{2}, & -1 \le c \le 1, \\ 0, & \text{else.} \end{cases}$$

In general, for $a \neq 0$, $f_Y(v) = \frac{1}{|a|} f_X\left(\frac{v-b}{a}\right)$. The negative scale factor mirrors the pdf: f_W decreases from 1 at $c = -1$ to 0 at $c = 1$.

(d) A server's response time T, in milliseconds, is exponentially distributed with mean 250 ms. Let S be the same response time in seconds. Find the pdf of S and identify its distribution.

Solution: T has parameter $\lambda = 1/250$ per ms. $S = T/1000$, so $a = 1/1000$ and $b = 0$. For $v \ge 0$,

$$f_S(v) = 1000\, f_T(1000v) = 1000 \cdot \frac{1}{250} e^{-1000v/250} = 4e^{-4v},$$

and $f_S(v) = 0$ for $v < 0$. So S is exponential with parameter 4 per second, with mean $1/4$ s $= 250$ ms as expected. The rate scales inversely with the time unit.

(e) More generally, let T_r have the Erlang distribution with parameters r and λ, and let $c > 0$. Use the scaling rule to show that cT_r has the Erlang distribution with parameters r and λ/c.

Solution: Let $Y = cT_r$. By the scaling rule with $a = c$, $b = 0$, for $v \ge 0$,

$$f_Y(v) = \frac{1}{c} f_{T_r}\left(\frac{v}{c}\right) = \frac{1}{c} \cdot \frac{\lambda^r (v/c)^{r-1} e^{-\lambda v/c}}{(r-1)!} = \frac{(\lambda/c)^r v^{r-1} e^{-(\lambda/c)v}}{(r-1)!},$$

which is the Erlang pdf with parameters r and λ/c. Part (d) is the special case $r = 1$. Taking $c = \lambda$ shows that λT_r always has the Erlang$(r, 1)$ distribution.

6. **[Gaussian distribution, Q and Φ functions]**
Let X have the $N(5, 16)$ distribution. Express each answer using Φ or Q, then give a numerical value.

(a) $P\{X > 11\}$

Solution: $P\{X > 11\} = P\left\{Z > \frac{11-5}{4}\right\} = Q(1.5) = 1 - \Phi(1.5) = 1 - 0.9332 = 0.0668.$

(b) $P\{1 < X \le 13\}$

Solution: Using $\Phi(-u) = 1 - \Phi(u)$:

$$\begin{aligned} P\{1 < X \le 13\} &= P\{-1 < Z \le 2\} = \Phi(2) - \Phi(-1) \\ &= \Phi(2) - (1 - \Phi(1)) = 0.9772 - 0.1587 = 0.8185. \end{aligned}$$

(c) $P\{|X - 5| > 8\}$

Solution: By symmetry of the standard normal pdf, $P\{|X - 5| > 8\} = P\{|Z| > 2\} = 2Q(2) \approx 2(0.02275) = 0.0455$. This matches the "about 95% within two standard deviations" rule from Section 3.6.2.

(d) $P\{X^2 > 81\}$

Solution: $\{X^2 > 81\} = \{X > 9\} \cup \{X < -9\}$, a union of disjoint events:

$$P\{X^2 > 81\} = P\{Z > 1\} + P\{Z < -3.5\} = Q(1) + Q(3.5) \approx 0.1587 + 2.33 \times 10^{-4} \approx 0.1589.$$

The left tail contributes very little, as in Example 3.6.4. A common mistake is to compute $P\{X > 9\}$ alone; the event $X < -9$ must also be included.

(e) Find c such that $P\{X > c\} = 0.05$.

Solution: We need $Q\left(\frac{c-5}{4}\right) = 0.05$. From the table, $Q(1.645) \approx 0.05$, so $\frac{c-5}{4} = 1.645$ and $c = 5 + 4(1.645) \approx 11.58$.

(f) Find a such that $P\{|X - 5| \leq a\} = 0.99$.

Solution: $P\{|X - 5| \leq a\} = 1 - 2Q(a/4) = 0.99$, so $Q(a/4) = 0.005$. From the table, $Q(2.576) \approx 0.005$, so $a = 4(2.576) \approx 10.30$.