

ECE 313: Problem Set 4: Problems and Solutions

Due: Friday, Oct. 2 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes*, Sections 2.4-2.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME

NETID

SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Maximum Likelihood, Poisson Distribution]**

The number of emails X a help desk receives per minute is modeled as Poisson with unknown mean $\lambda > 0$, so $P(X = k) = e^{-\lambda} \lambda^k / k!$ for $k = 0, 1, 2, \dots$. Suppose $X_1, \dots, X_n$ are i.i.d. observations.

- (a) Write down the likelihood function $L(\lambda)$ and the log-likelihood $\ell(\lambda)$.

Solution: Let $x_1, \dots, x_n$ denote the observed values. By independence,

$$L(\lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}.$$

Taking the natural logarithm,

$$\ell(\lambda) = \ln L(\lambda) = -n\lambda + \left(\sum_{i=1}^n x_i \right) \ln \lambda - \sum_{i=1}^n \ln(x_i!).$$

- (b) Derive the maximum likelihood estimator $\hat{\lambda}_{ML}$ and verify that it is a maximum.

Solution: Differentiating the log-likelihood,

$$\ell'(\lambda) = -n + \frac{1}{\lambda} \sum_{i=1}^n x_i.$$

Setting $\ell'(\lambda) = 0$ gives

$$\hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}.$$

The second derivative is

$$\ell''(\lambda) = -\frac{1}{\lambda^2} \sum_{i=1}^n x_i < 0$$

whenever $\sum_{i=1}^n x_i > 0$, so ℓ is strictly concave and $\hat{\lambda}_{ML} = \bar{x}$ is the global maximum.

(If all $x_i = 0$, then $L(\lambda) = e^{-n\lambda}$ is strictly decreasing, so the likelihood is maximized as $\lambda \rightarrow 0$, which again agrees with $\bar{x} = 0$.)

- (c) Over six one-minute intervals, the observed counts were 3, 1, 4, 2, 0, 2. Compute $\hat{\lambda}_{ML}$.

Solution: Here $n = 6$ and

$$\sum_{i=1}^6 x_i = 3 + 1 + 4 + 2 + 0 + 2 = 12.$$

Therefore,

$$\hat{\lambda}_{ML} = \frac{12}{6} = 2.$$

- (d) Using the invariance property of MLEs, estimate the probability that no emails arrive in a given minute.

Solution: Under the Poisson model,

$$P(X = 0) = e^{-\lambda}.$$

By the invariance property, the MLE of $g(\lambda) = e^{-\lambda}$ is $g(\hat{\lambda}_{ML})$. Hence

$$\hat{P}(X = 0) = e^{-\hat{\lambda}_{ML}} = e^{-2} \approx 0.135.$$

- (e) Is $\hat{\lambda}_{ML}$ unbiased? Compute its variance.

Solution: Since $E[X_i] = \lambda$ for each i , linearity of expectation gives

$$E[\hat{\lambda}_{ML}] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \cdot n\lambda = \lambda,$$

so $\hat{\lambda}_{ML}$ is unbiased.

Since the X_i are independent and $\text{Var}(X_i) = \lambda$,

$$\text{Var}(\hat{\lambda}_{ML}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n^2} \cdot n\lambda = \frac{\lambda}{n}.$$

2. [Maximum Likelihood, Geometric and Negative Binomial Distributions]

A basketball player repeatedly shoots free throws. Each shot is independently successful with unknown probability p .

- (a) Let X be the number of shots up to and including the first success, so $X \sim \text{Geometric}(p)$ with $P(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, \dots$. Given i.i.d. observations $X_1, \dots, X_n$, derive $\hat{p}_{ML}$.

Solution: Let $x_1, \dots, x_n$ denote the observed values and $s = \sum_{i=1}^n x_i$. By independence,

$$L(p) = \prod_{i=1}^n (1 - p)^{x_i-1} p = p^n (1 - p)^{s-n}.$$

The log-likelihood is

$$\ell(p) = n \ln p + (s - n) \ln(1 - p),$$

so

$$\ell'(p) = \frac{n}{p} - \frac{s - n}{1 - p}.$$

Setting $\ell'(p) = 0$ gives $n(1 - p) = p(s - n)$, i.e., $n = ps$. Therefore,

$$\hat{p}_{ML} = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}}.$$

Since $s \geq n$,

$$\ell''(p) = -\frac{n}{p^2} - \frac{s - n}{(1 - p)^2} < 0,$$

so this critical point is the maximum. (If every $x_i = 1$, then $L(p) = p^n$ is increasing and the maximum is at $p = 1$, which again agrees with $n/s = 1$.)

- (b) In five separate sessions, the numbers of shots until the first success were 2, 5, 1, 3, 4. Compute $\hat{p}_{ML}$.

Solution: Here $n = 5$ and

$$\sum_{i=1}^5 x_i = 2 + 5 + 1 + 3 + 4 = 15.$$

Therefore,

$$\hat{p}_{ML} = \frac{5}{15} = \frac{1}{3}.$$

- (c) Now suppose instead that the player shoots until the r -th success, and Y is the total number of shots taken, so $Y \sim \text{NegBin}(r, p)$ with

$$P(Y = y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}, \quad y = r, r+1, \dots$$

Derive $\hat{p}_{ML}$ based on a single observation $Y = y$, with r known.

Solution: The likelihood is

$$L(p) = \binom{y-1}{r-1} p^r (1-p)^{y-r},$$

and the log-likelihood is

$$\ell(p) = \ln \binom{y-1}{r-1} + r \ln p + (y-r) \ln(1-p).$$

The binomial coefficient does not depend on p , so

$$\ell'(p) = \frac{r}{p} - \frac{y-r}{1-p}.$$

Setting $\ell'(p) = 0$ gives $r(1-p) = p(y-r)$, i.e., $r = py$. Therefore,

$$\hat{p}_{ML} = \frac{r}{y}.$$

Since $y \geq r$,

$$\ell''(p) = -\frac{r}{p^2} - \frac{y-r}{(1-p)^2} < 0,$$

so this is the maximum.

- (d) If $r = 3$ and the player needed $y = 12$ shots, compute $\hat{p}_{ML}$.

Solution:

$$\hat{p}_{ML} = \frac{r}{y} = \frac{3}{12} = \frac{1}{4}.$$

- (e) Compare parts (a) and (c): how is the estimator in (c) related to the one in (a)? Give an intuitive explanation.

Solution: Both estimators have the same form:

$$\hat{p}_{ML} = \frac{\text{number of successes}}{\text{total number of shots}}.$$

In (a), there are n successes in $\sum_{i=1}^n x_i$ shots; in (c), there are r successes in y shots.

Intuitively, a negative binomial experiment is just r geometric experiments run back to back: $Y = X_1 + \cdots + X_r$, where X_j is the number of shots between the $(j-1)$ -th and j -th successes. Setting $n = r$ and $\sum_{i=1}^n x_i = y$, the geometric likelihood $p^n(1-p)^{s-n}$ and the negative binomial likelihood $\binom{y-1}{r-1} p^r(1-p)^{y-r}$ differ only by a constant that does not depend on p . Hence they are maximized at the same value, and both estimators reduce to the observed fraction of successful shots.

3. [Markov and Chebyshev Inequalities]

The number of customers X arriving at a small shop in an hour is a nonnegative integer-valued random variable with $E[X] = 4$.

- (a) Prove Markov's inequality for a nonnegative discrete random variable: $P(X \geq a) \leq E[X]/a$ for any $a > 0$. (*Hint:* write $E[X] = \sum_k k P(X = k)$ and drop the terms with $k < a$.)

Solution: Since X is nonnegative, every term $k P(X = k)$ in the sum is nonnegative. Dropping the terms with $k < a$ can therefore only decrease the sum:

$$E[X] = \sum_k k P(X = k) \geq \sum_{k \geq a} k P(X = k).$$

For each remaining term, $k \geq a$, so

$$\sum_{k \geq a} k P(X = k) \geq \sum_{k \geq a} a P(X = k) = a P(X \geq a).$$

Combining these and dividing by $a > 0$,

$$P(X \geq a) \leq \frac{E[X]}{a}.$$

- (b) Using only the mean, bound $P(X \geq 10)$.

Solution: By Markov's inequality with $a = 10$,

$$P(X \geq 10) \leq \frac{E[X]}{10} = \frac{4}{10} = 0.4.$$

- (c) Now suppose you also know $X \sim \text{Poisson}(4)$. Use Chebyshev's inequality to bound $P(X \geq 10)$.

Solution: For $X \sim \text{Poisson}(4)$,

$$E[X] = 4, \quad \text{Var}(X) = 4.$$

If $X \geq 10$, then $X - 4 \geq 6$, so $|X - 4| \geq 6$. Hence

$$P(X \geq 10) \leq P(|X - 4| \geq 6) \leq \frac{\text{Var}(X)}{6^2} = \frac{4}{36} = \frac{1}{9} \approx 0.111.$$

- (d) Compute the exact value of $P(X \geq 10)$ under the Poisson model and compare it with the bounds from (b) and (c).

Solution: Using the complement,

$$P(X \geq 10) = 1 - \sum_{k=0}^9 e^{-4} \frac{4^k}{k!}.$$

Evaluating the sum,

$$\sum_{k=0}^9 \frac{4^k}{k!} = 1 + 4 + 8 + \frac{32}{3} + \frac{32}{3} + \frac{128}{15} + \frac{256}{45} + \frac{1024}{315} + \frac{512}{315} + \frac{2048}{2835} \approx 54.154,$$

so

$$P(X \leq 9) \approx e^{-4}(54.154) \approx 0.99187,$$

and

$$P(X \geq 10) \approx 1 - 0.99187 = 0.0081.$$

All three values are consistent:

$$0.0081 \leq 0.111 \leq 0.4.$$

Both bounds are valid but loose. The Chebyshev bound is tighter than the Markov bound because it uses the variance in addition to the mean, but it is still more than ten times the exact value, because it uses only two moments and ignores the actual shape of the Poisson distribution.

- (e) Construct a nonnegative integer-valued random variable with mean 4 for which the Markov bound in part (b) holds with equality.

Solution: Equality in the proof of part (a) requires that no probability be lost in either inequality: all of the mass with $k < 10$ must sit at $k = 0$, and all of the mass with $k \geq 10$ must sit at exactly $k = 10$. So let

$$X = \begin{cases} 0, & \text{with probability } 0.6, \\ 10, & \text{with probability } 0.4. \end{cases}$$

Then X is nonnegative and integer-valued, with

$$E[X] = 0(0.6) + 10(0.4) = 4,$$

and

$$P(X \geq 10) = 0.4 = \frac{E[X]}{10}.$$

Hence the Markov bound holds with equality.

4. **[Chebyshev's Inequality and the Law of Large Numbers]**

A fair coin is flipped n times, and X denotes the number of heads, so $X \sim \text{Binomial}(n, 1/2)$.

- (a) For $n = 100$, use Chebyshev's inequality to find a lower bound on $P(41 \leq X \leq 59)$.

Solution: Since $X \sim \text{Binomial}(100, 1/2)$,

$$E[X] = np = 50$$

and

$$\text{Var}(X) = np(1 - p) = 25.$$

Notice that

$$P(41 \leq X \leq 59) = P(|X - 50| < 10).$$

By Chebyshev's inequality,

$$P(|X - 50| \geq 10) \leq \frac{\text{Var}(X)}{10^2} = \frac{25}{100} = \frac{1}{4}.$$

Therefore,

$$P(41 \leq X \leq 59) \geq 1 - \frac{1}{4} = \frac{3}{4}.$$

- (b) For $n = 100$, bound $P(X \geq 60)$ using Markov's inequality and then using Chebyshev's inequality. Which is tighter, and why?

Solution: Since X is nonnegative and $E[X] = 50$, Markov's inequality gives

$$P(X \geq 60) \leq \frac{E[X]}{60} = \frac{50}{60} = \frac{5}{6} \approx 0.833.$$

For Chebyshev's inequality, if $X \geq 60$, then $|X - 50| \geq 10$. Hence

$$P(X \geq 60) \leq P(|X - 50| \geq 10) \leq \frac{\text{Var}(X)}{10^2} = \frac{25}{100} = \frac{1}{4}.$$

Thus, Chebyshev's inequality gives the tighter bound. Markov's inequality uses only the mean of X , whereas Chebyshev's inequality also uses the variance and the fact that X is concentrated around its mean.

- (c) For general n , show that for any $\varepsilon > 0$,

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \leq \frac{1}{4n\varepsilon^2},$$

and explain what happens as $n \rightarrow \infty$.

Solution: For $X \sim \text{Binomial}(n, 1/2)$,

$$E[X] = \frac{n}{2}, \quad \text{Var}(X) = \frac{n}{4}.$$

We have

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| \geq \varepsilon\right) = P\left(\left|X - \frac{n}{2}\right| \geq n\varepsilon\right).$$

Applying Chebyshev's inequality,

$$P\left(\left|X - \frac{n}{2}\right| \geq n\varepsilon\right) \leq \frac{\text{Var}(X)}{n^2\varepsilon^2}.$$

Therefore,

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \leq \frac{n/4}{n^2\varepsilon^2} = \frac{1}{4n\varepsilon^2}.$$

For every fixed $\varepsilon > 0$,

$$\frac{1}{4n\varepsilon^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus,

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \rightarrow 0,$$

which is an instance of the weak law of large numbers.

- (d) How many flips are needed to guarantee, by Chebyshev's inequality, that the fraction of heads is within 0.02 of 1/2 with probability at least 0.95?

Solution: We want

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| < 0.02\right) \geq 0.95.$$

Equivalently, it is sufficient to require

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| \geq 0.02\right) \leq 0.05.$$

Using the bound from part (c),

$$\frac{1}{4n(0.02)^2} \leq 0.05.$$

Since

$$4(0.02)^2 = 0.0016,$$

we obtain

$$\frac{1}{0.0016n} \leq 0.05,$$

or

$$n \geq \frac{1}{0.0016(0.05)} = 12500.$$

Therefore, at least

$$12500$$

flips are needed.

5. [Binary Hypothesis Testing, Poisson Observations]

A Geiger counter records X clicks in one second. If no radioactive source is present (H_0), $X \sim \text{Poisson}(2)$. If a source is present (H_1), $X \sim \text{Poisson}(5)$.

- (a) Write the likelihood ratio $\Lambda(k) = P(X = k | H_1)/P(X = k | H_0)$ and show that any likelihood ratio test reduces to a rule of the form “decide H_1 if $X \geq k^*$ ” for some integer k^* .

Solution: Under the two hypotheses,

$$P(X = k | H_0) = e^{-2} \frac{2^k}{k!}$$

and

$$P(X = k | H_1) = e^{-5} \frac{5^k}{k!}.$$

Therefore, the likelihood ratio is

$$\Lambda(k) = \frac{P(X = k | H_1)}{P(X = k | H_0)} = \frac{e^{-5} 5^k / k!}{e^{-2} 2^k / k!} = e^{-3} \left(\frac{5}{2} \right)^k.$$

Since $5/2 > 1$, $\Lambda(k)$ is strictly increasing in k . Thus, for any likelihood ratio test of the form

$$\Lambda(k) \geq \eta,$$

there is an integer threshold k^* such that the test is equivalent to

$$\text{decide } H_1 \text{ if } X \geq k^*.$$

- (b) Find the maximum likelihood (ML) decision rule. Compute the probability of false alarm $P_{FA} = P(\text{decide } H_1 | H_0)$ and the probability of miss $P_M = P(\text{decide } H_0 | H_1)$.

Solution: The ML rule chooses the hypothesis under which the observed value has the larger likelihood. Thus, we decide H_1 when

$$\Lambda(k) \geq 1.$$

Using the likelihood ratio from part (a),

$$e^{-3} \left(\frac{5}{2} \right)^k \geq 1,$$

which is equivalent to

$$k \geq \frac{3}{\ln(5/2)} \approx 3.27.$$

Since k is an integer, the ML decision rule is

$$\text{decide } H_1 \text{ if } X \geq 4.$$

The probability of false alarm is

$$P_{FA} = P(X \geq 4 | H_0) = 1 - P(X \leq 3 | H_0).$$

Since $X \sim \text{Poisson}(2)$ under H_0 ,

$$P_{FA} = 1 - e^{-2} \sum_{k=0}^3 \frac{2^k}{k!} \approx 0.143.$$

The probability of miss is

$$P_M = P(X \leq 3 \mid H_1).$$

Since $X \sim \text{Poisson}(5)$ under H_1 ,

$$P_M = e^{-5} \sum_{k=0}^3 \frac{5^k}{k!} \approx 0.265.$$

- (c) Suppose the prior probabilities are $P(H_0) = 0.7$ and $P(H_1) = 0.3$. Find the MAP decision rule and its overall probability of error. Compare it with the error probability of the ML rule under these same priors.

Solution: The MAP rule decides H_1 when

$$P(H_1)P(X = k \mid H_1) \geq P(H_0)P(X = k \mid H_0).$$

Equivalently,

$$\Lambda(k) \geq \frac{P(H_0)}{P(H_1)} = \frac{0.7}{0.3} = \frac{7}{3}.$$

Thus,

$$e^{-3} \left(\frac{5}{2}\right)^k \geq \frac{7}{3},$$

or

$$k \geq \frac{3 + \ln(7/3)}{\ln(5/2)} \approx 4.20.$$

Therefore, the MAP rule is

$$\text{decide } H_1 \text{ if } X \geq 5.$$

For this rule,

$$P_{FA} = P(X \geq 5 \mid H_0) = 1 - e^{-2} \sum_{k=0}^4 \frac{2^k}{k!} \approx 0.053,$$

and

$$P_M = P(X \leq 4 \mid H_1) = e^{-5} \sum_{k=0}^4 \frac{5^k}{k!} \approx 0.440.$$

Hence the overall probability of error is

$$P_e = P(H_0)P_{FA} + P(H_1)P_M,$$

so

$$P_e \approx 0.7(0.053) + 0.3(0.440) \approx 0.169.$$

For comparison, the ML rule from part (b) has

$$P_e^{ML} = 0.7(0.143) + 0.3(0.265) \approx 0.180.$$

Thus, under these prior probabilities, the MAP rule has a smaller probability of error than the ML rule.

- (d) Suppose we require $P_{FA} \leq 0.05$. Find the smallest integer threshold k^* satisfying this, and compute the resulting probability of detection $P_D = 1 - P_M$. Why can't we achieve $P_{FA} = 0.05$ exactly with a deterministic threshold rule?

Solution: For a threshold rule that decides H_1 when $X \geq k^*$,

$$P_{FA} = P(X \geq k^* | H_0).$$

For $k^* = 5$,

$$P_{FA} = P(\text{Poisson}(2) \geq 5) \approx 0.053,$$

which is greater than 0.05.

For $k^* = 6$,

$$P_{FA} = P(\text{Poisson}(2) \geq 6) = 1 - e^{-2} \sum_{k=0}^5 \frac{2^k}{k!} \approx 0.017.$$

Therefore, the smallest threshold satisfying the constraint is

$$k^* = 6.$$

The probability of detection is

$$P_D = P(X \geq 6 | H_1).$$

Under H_1 , $X \sim \text{Poisson}(5)$, so

$$P_D = 1 - e^{-5} \sum_{k=0}^5 \frac{5^k}{k!} \approx 0.384.$$

Because X is discrete, changing an integer threshold changes P_{FA} in discrete jumps. In particular, the false alarm probability jumps from approximately 0.053 at threshold 5 to approximately 0.017 at threshold 6. Therefore, a deterministic threshold rule cannot achieve $P_{FA} = 0.05$ exactly. Achieving exactly 0.05 would require randomization at the boundary.

6. [Confidence Intervals and Confidence Level]

A survey of $n = 400$ randomly selected students finds that 240 support a new campus policy. Model each response as $X_i \sim \text{Bernoulli}(p)$, independent, and let $\hat{p} = \frac{1}{n} \sum_{i=1}^n X_i$.

- (a) Show that $\hat{p}$ is the maximum likelihood estimator of p , and compute its value.

Solution: Let $S = \sum_{i=1}^n X_i$. Since the observations are independent Bernoulli random variables, the likelihood function is

$$L(p) = \prod_{i=1}^n p^{X_i} (1-p)^{1-X_i} = p^S (1-p)^{n-S}.$$

The log-likelihood is

$$\ell(p) = S \ln p + (n - S) \ln(1 - p).$$

Differentiating,

$$\ell'(p) = \frac{S}{p} - \frac{n-S}{1-p}.$$

Setting $\ell'(p) = 0$ gives $S(1-p) = p(n-S)$, so $S = np$. Hence,

$$\hat{p}_{ML} = \frac{S}{n} = \frac{1}{n} \sum_{i=1}^n X_i = \hat{p}.$$

Also,

$$\ell''(p) = -\frac{S}{p^2} - \frac{n-S}{(1-p)^2} < 0,$$

so the critical point is a maximum. For the observed data,

$$\hat{p} = \frac{240}{400} = 0.6.$$

- (b) Using Chebyshev's inequality and $p(1-p) \leq 1/4$, show that $P(\hat{p} - \varepsilon < p < \hat{p} + \varepsilon) \geq 1 - \frac{1}{4n\varepsilon^2}$.

Solution: Since $\hat{p} = \frac{1}{n} \sum_{i=1}^n X_i$ and the X_i are independent Bernoulli(p) random variables,

$$E[\hat{p}] = p, \quad \text{Var}(\hat{p}) = \frac{p(1-p)}{n} \leq \frac{1}{4n}.$$

By Chebyshev's inequality,

$$P(|\hat{p} - p| \geq \varepsilon) \leq \frac{\text{Var}(\hat{p})}{\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}.$$

Therefore,

$$P(|\hat{p} - p| < \varepsilon) \geq 1 - \frac{1}{4n\varepsilon^2}.$$

Since $|\hat{p} - p| < \varepsilon$ is equivalent to $\hat{p} - \varepsilon < p < \hat{p} + \varepsilon$,

$$P(\hat{p} - \varepsilon < p < \hat{p} + \varepsilon) \geq 1 - \frac{1}{4n\varepsilon^2}.$$

- (c) Find ε such that $[\hat{p} - \varepsilon, \hat{p} + \varepsilon]$ is a confidence interval for p with confidence level at least 95%. Compute the interval for this data.

Solution: For confidence level at least 95%, we require

$$1 - \frac{1}{4n\varepsilon^2} \geq 0.95.$$

With $n = 400$,

$$\frac{1}{4(400)\varepsilon^2} \leq 0.05.$$

Thus,

$$\varepsilon \geq \frac{1}{\sqrt{80}} \approx 0.112.$$

Taking $\varepsilon = 1/\sqrt{80}$ gives

$$[\hat{p} - \varepsilon, \hat{p} + \varepsilon] = [0.488, 0.712].$$

- (d) Repeat part (c) for a confidence level of at least 90%. Explain the trade-off between the confidence level and the width of the interval.

Solution: For confidence level at least 90%, we require

$$1 - \frac{1}{4n\varepsilon^2} \geq 0.90.$$

With $n = 400$,

$$\frac{1}{4(400)\varepsilon^2} \leq 0.10.$$

Thus,

$$\varepsilon \geq \frac{1}{\sqrt{160}} \approx 0.079.$$

Taking $\varepsilon = 1/\sqrt{160}$ gives

$$[\hat{p} - \varepsilon, \hat{p} + \varepsilon] = [0.521, 0.679].$$

A lower confidence level gives a narrower interval, while a higher confidence level requires a wider interval.

- (e) A classmate says, “There is a 95% probability that p lies in the interval from part (c).” Explain what is wrong with this statement and give the correct interpretation of a confidence level.

Solution: The parameter p is fixed, while the confidence interval is random because it depends on the random sample. Once the data are observed, the resulting interval either contains p or it does not.

The correct interpretation is that if the sampling procedure were repeated many times and an interval were constructed in the same way each time, at least 95% of those intervals would contain the true value of p .

- (f) How large must n be for a 95% (Chebyshev-based) confidence interval to have half-width at most 0.05?

Solution: For a 95% confidence level with half-width $\varepsilon = 0.05$, we require

$$\frac{1}{4n(0.05)^2} \leq 0.05.$$

Therefore,

$$n \geq \frac{1}{4(0.05)^2(0.05)} = 2000.$$

Thus, at least 2000 students are needed.