

ECE 313: Problem Set 4

Due: Friday, Oct. 2 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes*, Sections 2.4-2.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME

NETID

SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Maximum Likelihood, Poisson Distribution]**

The number of emails X a help desk receives per minute is modeled as Poisson with unknown mean $\lambda > 0$, so $P(X = k) = e^{-\lambda} \lambda^k / k!$ for $k = 0, 1, 2, \dots$. Suppose $X_1, \dots, X_n$ are i.i.d. observations.

- (a) Write down the likelihood function $L(\lambda)$ and the log-likelihood $\ell(\lambda)$.
- (b) Derive the maximum likelihood estimator $\hat{\lambda}_{ML}$ and verify that it is a maximum.
- (c) Over six one-minute intervals, the observed counts were 3, 1, 4, 2, 0, 2. Compute $\hat{\lambda}_{ML}$.
- (d) Using the invariance property of MLEs, estimate the probability that no emails arrive in a given minute.
- (e) Is $\hat{\lambda}_{ML}$ unbiased? Compute its variance.

2. **[Maximum Likelihood, Geometric and Negative Binomial Distributions]**

A basketball player repeatedly shoots free throws. Each shot is independently successful with unknown probability p .

- (a) Let X be the number of shots up to and including the first success, so $X \sim \text{Geometric}(p)$ with $P(X = k) = (1 - p)^{k-1} p$ for $k = 1, 2, \dots$. Given i.i.d. observations $X_1, \dots, X_n$, derive $\hat{p}_{ML}$.
- (b) In five separate sessions, the numbers of shots until the first success were 2, 5, 1, 3, 4. Compute $\hat{p}_{ML}$.
- (c) Now suppose instead that the player shoots until the r -th success, and Y is the total

number of shots taken, so $Y \sim \text{NegBin}(r, p)$ with

$$P(Y = y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}, \quad y = r, r+1, \dots$$

Derive $\hat{p}_{ML}$ based on a single observation $Y = y$, with r known.

- (d) If $r = 3$ and the player needed $y = 12$ shots, compute $\hat{p}_{ML}$.
- (e) Compare parts (a) and (c): how is the estimator in (c) related to the one in (a)? Give an intuitive explanation.

3. [Markov and Chebyshev Inequalities]

The number of customers X arriving at a small shop in an hour is a nonnegative integer-valued random variable with $E[X] = 4$.

- (a) Prove Markov's inequality for a nonnegative discrete random variable: $P(X \geq a) \leq E[X]/a$ for any $a > 0$. (*Hint*: write $E[X] = \sum_k k P(X = k)$ and drop the terms with $k < a$.)
- (b) Using only the mean, bound $P(X \geq 10)$.
- (c) Now suppose you also know $X \sim \text{Poisson}(4)$. Use Chebyshev's inequality to bound $P(X \geq 10)$.
- (d) Compute the exact value of $P(X \geq 10)$ under the Poisson model and compare it with the bounds from (b) and (c).
- (e) Construct a nonnegative integer-valued random variable with mean 4 for which the Markov bound in part (b) holds with equality.

4. [Chebyshev's Inequality and the Law of Large Numbers]

A fair coin is flipped n times, and X denotes the number of heads, so $X \sim \text{Binomial}(n, 1/2)$.

- (a) For $n = 100$, use Chebyshev's inequality to find a lower bound on $P(41 \leq X \leq 59)$.
- (b) For $n = 100$, bound $P(X \geq 60)$ using Markov's inequality and then using Chebyshev's inequality. Which is tighter, and why?
- (c) For general n , show that for any $\varepsilon > 0$,

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \leq \frac{1}{4n\varepsilon^2},$$

and explain what happens as $n \rightarrow \infty$.

- (d) How many flips are needed to guarantee, by Chebyshev's inequality, that the fraction of heads is within 0.02 of 1/2 with probability at least 0.95?

5. [Binary Hypothesis Testing, Poisson Observations]

A Geiger counter records X clicks in one second. If no radioactive source is present (H_0), $X \sim \text{Poisson}(2)$. If a source is present (H_1), $X \sim \text{Poisson}(5)$.

- (a) Write the likelihood ratio $\Lambda(k) = P(X = k | H_1)/P(X = k | H_0)$ and show that any likelihood ratio test reduces to a rule of the form "decide H_1 if $X \geq k^*$ " for some integer k^* .
- (b) Find the maximum likelihood (ML) decision rule. Compute the probability of false alarm $P_{FA} = P(\text{decide } H_1 | H_0)$ and the probability of miss $P_M = P(\text{decide } H_0 | H_1)$.

- (c) Suppose the prior probabilities are $P(H_0) = 0.7$ and $P(H_1) = 0.3$. Find the MAP decision rule and its overall probability of error. Compare it with the error probability of the ML rule under these same priors.
- (d) Suppose we require $P_{FA} \leq 0.05$. Find the smallest integer threshold k^* satisfying this, and compute the resulting probability of detection $P_D = 1 - P_M$. Why can't we achieve $P_{FA} = 0.05$ exactly with a deterministic threshold rule?

6. **[Confidence Intervals and Confidence Level]**

A survey of $n = 400$ randomly selected students finds that 240 support a new campus policy. Model each response as $X_i \sim \text{Bernoulli}(p)$, independent, and let $\hat{p} = \frac{1}{n} \sum_{i=1}^n X_i$.

- (a) Show that $\hat{p}$ is the maximum likelihood estimator of p , and compute its value.
- (b) Using Chebyshev's inequality and $p(1-p) \leq 1/4$, show that $P(\hat{p} - \varepsilon < p < \hat{p} + \varepsilon) \geq 1 - \frac{1}{4n\varepsilon^2}$.
- (c) Find ε such that $[\hat{p} - \varepsilon, \hat{p} + \varepsilon]$ is a confidence interval for p with confidence level at least 95%. Compute the interval for this data.
- (d) Repeat part (c) for a confidence level of at least 90%. Explain the trade-off between the confidence level and the width of the interval.
- (e) A classmate says, "There is a 95% probability that p lies in the interval from part (c)." Explain what is wrong with this statement and give the correct interpretation of a confidence level.
- (f) How large must n be for a 95% (Chebyshev-based) confidence interval to have half-width at most 0.05?