

ECE 313: Problem Set 3: Problems and Solutions

Due: Friday, September 25 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes, Section 2.3, 2.10, 2.4.1-2.4.2*

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME

NETID

SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Conditional probability and the binomial distribution]**

Let X denote the number of heads observed in 12 tosses of a fair coin.

- (a) Find $P\{X \leq 6 \mid X \geq 5\}$.

Solution:

$$\begin{aligned} P\{X \leq 6 \mid X \geq 5\} &= \frac{P\{X \leq 6, X \geq 5\}}{P\{X \geq 5\}} = \frac{P\{X \in \{5, 6\}\}}{1 - P\{X < 5\}} \\ &= \frac{\binom{12}{5} \left(\frac{1}{2}\right)^{12} + \binom{12}{6} \left(\frac{1}{2}\right)^{12}}{1 - \left[\binom{12}{0} \left(\frac{1}{2}\right)^{12} + \binom{12}{1} \left(\frac{1}{2}\right)^{12} + \binom{12}{2} \left(\frac{1}{2}\right)^{12} + \binom{12}{3} \left(\frac{1}{2}\right)^{12} + \binom{12}{4} \left(\frac{1}{2}\right)^{12}\right]} \\ &= \frac{66}{127} \approx 0.5197 \end{aligned}$$

because X is a binomial random variable with parameters $n = 12$ and $p = 1/2$.

- (b) Given that $X = 5$, what is the (conditional) probability that the 5th toss was a head?

Solution: Let H_5 denote the event that the fifth toss shows heads, and let $\hat{X}$ be the number of heads observed in the other eleven tosses. Then, the events H_5 and $\{\hat{X} = 4\}$ are independent (because the tosses are independent), and $\hat{X}$ is a binomial random variable with parameters $n = 11$ and $p = 1/2$. Therefore,

$$P\{H_5 \mid X = 5\} = \frac{P\{H_5, X = 5\}}{P\{X = 5\}} = \frac{P\{H_5, \hat{X} = 4\}}{P\{X = 5\}} = \frac{P(H_5) P\{\hat{X} = 4\}}{P\{X = 5\}}$$

$$= \frac{\frac{1}{2} \binom{11}{4} \left(\frac{1}{2}\right)^{11}}{\binom{12}{5} \left(\frac{1}{2}\right)^{12}} = \frac{5}{12}$$

2. [Independence]

Suppose two dice, one orange and one blue, are rolled. Define the following events:

- A*: The product of the two numbers that show is 8.
- B*: The number on the orange die is strictly larger than the number on the blue die.
- C*: The sum of the numbers is divisible by four.
- D*: The number on the orange die is either 2 or 5.

- (a) List all pairs of events from the set A, B, C, D that are independent.

Solution: For brevity, we write ij for the outcome that the orange die shows i and the blue die shows j . Then,

$$\begin{aligned} A &= \{24, 42\} \\ B &= \{ij : 1 \leq j < i \leq 6\} \\ C &= \{13, 22, 31, 26, 35, 44, 53, 66\} \\ D &= \{2j : 1 \leq j \leq 6\} \cup \{5j : 1 \leq j \leq 6\} \end{aligned}$$

$$P(A) = \frac{1}{18}, \quad P(B) = \frac{5}{12}, \quad P(C) = \frac{1}{4}, \quad P(D) = \frac{1}{3}$$

$$\begin{aligned} P(AB) &= P\{42\} = \frac{1}{36}, & P(AC) &= P(\emptyset) = 0, & P(AD) &= P\{24\} = \frac{1}{36} \\ P(BC) &= P\{31, 53, 62\} = \frac{1}{12}, & P(BD) &= P\{21, 51, 52, 53, 54\} = \frac{5}{36}, & P(CD) &= P\{22, 26, 53\} = \frac{1}{12} \end{aligned}$$

Comparing each intersection probability with the product of the marginal probabilities:

$$\begin{aligned} P(A)P(B) &= \frac{5}{216} \neq \frac{1}{36} = P(AB) & P(A)P(C) &= \frac{1}{72} \neq 0 = P(AC) \\ P(A)P(D) &= \frac{1}{54} \neq \frac{1}{36} = P(AD) & P(B)P(C) &= \frac{5}{48} \neq \frac{1}{12} = P(BC) \\ P(B)P(D) &= \frac{5}{36} = P(BD) & P(C)P(D) &= \frac{1}{12} = P(CD) \end{aligned}$$

So B and D are independent, and C and D are independent; no other pair is independent.

- (b) List all triplets of events, if any, from the set of A, B, C, D , that are mutually independent.

Solution: By part (a), the only independent pairs are $\{B, D\}$ and $\{C, D\}$; every other pair is dependent. Since mutual independence of a triplet requires all three of its pairs to be independent, and every triplet drawn from $\{A, B, C, D\}$ contains at least one dependent pair (ABC , ABD , and ACD each contain AB , AC , or AD ; BCD contains BC), no three events are mutually independent.

3. [Binomial distribution]

Six cars start out on a cross-country race. The probability that a car breaks down and drops out of the race is 0.1. Cars break down independently of each other.

- (a) What is the probability that exactly three cars finish the race?

Solution: Let X denote the number of cars that finish the race. Thus, X has the binomial distribution with parameters $n = 6$ and $p = 0.9$. Then,

$$P\{X = 3\} = \binom{6}{3}(0.9)^3(0.1)^3 = \frac{14580}{1000000} = 0.01458.$$

- (b) What is the probability that at most three cars finish the race?

Solution:

$$P\{X \leq 3\} = \sum_{i=0}^3 \binom{6}{i}(0.9)^i(0.1)^{6-i} = \frac{1 + 54 + 1215 + 14580}{1000000} = \frac{15850}{1000000} = 0.01585.$$

- (c) What is the probability that at least four cars finish the race?

Solution:

$$P\{X \geq 4\} = \sum_{i=4}^6 \binom{6}{i}(0.9)^i(0.1)^{6-i} = 1 - P\{X \leq 3\} = 1 - 0.01585 = 0.98415.$$

4. [Geometric Distribution]

Let X be a geometric random variable with parameter $p = 1/3$, and let $Y = \cos\left(\frac{\pi X}{2}\right)$. Find the pmf of Y in closed-form.

Solution: Since $X \in \{1, 2, \dots\}$, Y has a value in $\{-1, 0, 1\}$. Explicitly,

$$Y = \begin{cases} 0, & \text{if } X = 2k + 1, & k \in \mathbb{Z}, k \geq 0 \\ 1, & \text{if } X = 4k, & k \in \mathbb{Z}, k \geq 1 \\ -1, & \text{if } X = 4k + 2, & k \in \mathbb{Z}, k \geq 0 \end{cases}.$$

Hence,

$$\begin{aligned} P(Y = 0) &= P(X \in \{2k + 1 : k \in \mathbb{Z}, k \geq 0\}) \\ &= \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{2k} \left(\frac{1}{3}\right) = \left(\frac{1}{3}\right) \sum_{k=0}^{\infty} \left(\frac{4}{9}\right)^k \\ &= \left(\frac{1}{3}\right) \cdot \frac{1}{1 - \frac{4}{9}} = \frac{3}{5} = \frac{39}{65} \end{aligned}$$

$$\begin{aligned} P(Y = 1) &= P(X \in \{4k : k \in \mathbb{Z}, k \geq 1\}) \\ &= \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{4k-1} \left(\frac{1}{3}\right) = \left(\frac{8}{81}\right) \sum_{k=0}^{\infty} \left(\frac{16}{81}\right)^k \\ &= \left(\frac{8}{81}\right) \cdot \frac{1}{1 - \frac{16}{81}} = \frac{8}{65} \end{aligned}$$

$$\begin{aligned} P(Y = -1) &= P(X \in \{4k + 2 : k \in \mathbb{Z}, k \geq 0\}) \\ &= \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{4k+1} \left(\frac{1}{3}\right) = \left(\frac{2}{9}\right) \sum_{k=0}^{\infty} \left(\frac{16}{81}\right)^k \\ &= \left(\frac{2}{9}\right) \cdot \frac{1}{1 - \frac{16}{81}} = \frac{18}{65} \end{aligned}$$

5. **[Negative Binomial Distribution]**

Let X be a random variable representing the number of times one must throw a red die until the outcome 2 has occurred five times. Let Y be a random variable representing the number of times one must throw a blue die until the outcomes 1 or 2 have occurred five times.

- (a) Find the pmf, the expected value and the standard deviation of X .

Solution: The experiment consists of a sequence of independent and identically distributed Bernoulli experiments and the random variable X denotes the number of times until a certain event, namely the event that the die shows 2, has occurred a certain number of times, namely five times. Hence, X has a negative binomial distribution with parameters $r = 5$ and $p = 1/6$, where p is the probability that 2 shows. From page 45 of the textbook, we have

$$P(X = n) = \binom{n-1}{4} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^{n-5}, \quad n \geq 5 \quad (1)$$

Furthermore,

- $E[X] = \frac{r}{p} = 30$
- $\text{Var}(X) = \frac{r(1-p)}{p^2} = 150$
- $\sigma = \sqrt{\text{Var}(X)} = \sqrt{150}$

- (b) Find the pmf, the expected value, and the standard deviation of Y .

Solution: For this part, $p = \frac{1}{3}$ and $r = 5$. pmf of Y is given by

$$P(Y = n) = \binom{n-1}{4} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{n-5}, \quad n \geq 5 \quad (2)$$

Additionally,

- $E[Y] = \frac{r}{p} = 15$
- $\text{Var}(Y) = \frac{r(1-p)}{p^2} = 30$
- $\sigma = \sqrt{\text{Var}(Y)} = \sqrt{30}$

6. **[Flipping a coin]**

Suppose that a fair coin is independently tossed twice. Define the events:

$$\begin{aligned} A &= \{\text{"The first toss is a head"}\} \\ B &= \{\text{"The second toss is a head"}\} \\ C &= \{\text{"Both tosses yield the same outcome"}\} \end{aligned}$$

Are A, B, C independent? Please show your work.

Solution: The sample space is:

$$\Omega = \{HH, HT, TH, TT\}$$

Also, $A = \{HH, HT\}$, $B = \{HH, TH\}$, and $C = \{HH, TT\}$. Since the coin is fair:

$$P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$$

We have: $A \cap B = \{HH\}$, $A \cap C = \{HH\}$ and $B \cap C = \{HH\}$.

Therefore,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4} = P(A)P(B) = P(A)P(C) = P(B)P(C),$$

i.e., A, B, C are pairwise independent.

We have $A \cap B \cap C = \{HH\}$, i.e.,

$$P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$$

Therefore, A, B and C are not independent.