Caveats

- Problems in the slide will be independent from midterm problems
  - $P(p_1 | p_1' \text{ in slide}) = P(p_1)$
- All numbers will be replaced by symbols in the slide
  - In midterm, you may need to compute
- We will cover top-K options from the Slido survey
  - Survey does not cover all topics
  - You still need to review all topics by yourself

Agenda & Survey result

- Poisson Process
  - Exponential Distribution
- Scaling of PDF
- Markov and Chebyshev
- PDF/CDF

Bernoulli Process $\xrightarrow{h \to 0, \lambda = \frac{p}{h}}$ Poisson Process

- Assume each trial takes h duration to complete

discrete

$Bern(p)$

$Bin\left(n = \left\lfloor \frac{t}{h} \right\rfloor, p = \lambda h\right)$

$Geo(p = \lambda h)$

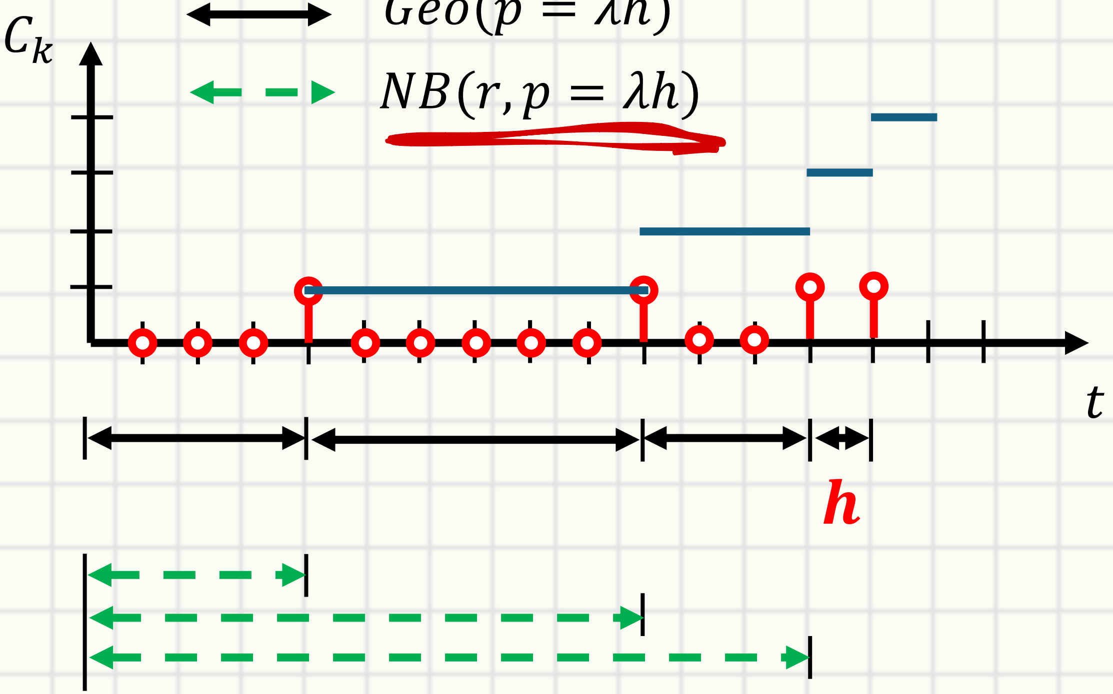
$NB(r, p = \lambda h)$

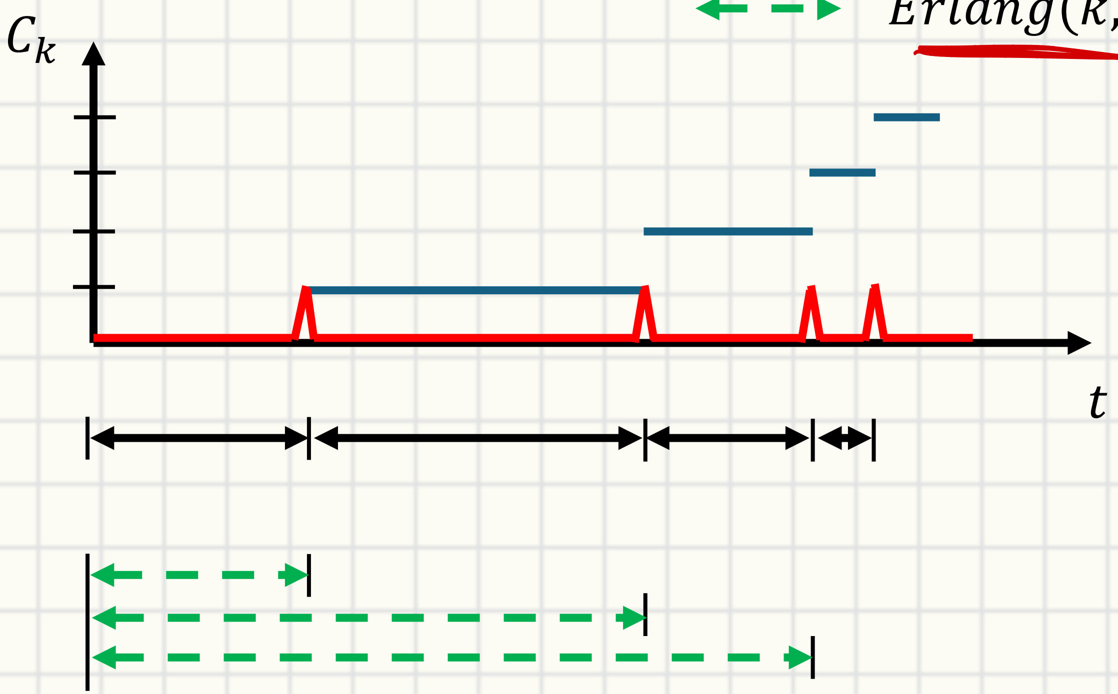
$$\xrightarrow{h \to 0, \lambda = \frac{p}{h} = \left(\frac{P(Head)}{Time}\right)}$$

$Poi(\lambda = \lambda t)$

$Exp(\lambda)$

$Erlang(k, \lambda)$





Properties

CCDF $e^{-\lambda t}$

λ of Poisson process

	$Exp(\lambda)$	$Poi(\lambda = \lambda_{ref} t)$
Mean	$\frac{1}{\lambda}$.	λ
Variance	$\frac{1}{\lambda^2}$	λ
PDF/ PMF	$f_X(t) = \lambda e^{-\lambda t}$.	$P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$
CDF	$1 - e^{-\lambda t}$	$\sum_{k=0}^{m} \frac{\lambda^k e^{-\lambda}}{k!}$
Example	System lifetime	Event occurrence within t
Special	Memoryless	Memoryless

Poisson Process

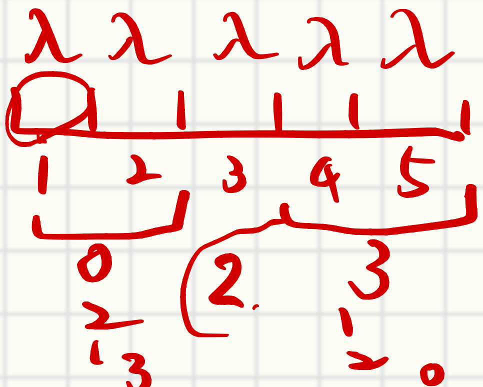
Poisson RV.

- $\lambda = \lambda_{ref} t$
- None-overlapped process are independent.

A support center is receiving λ call/ mins. Probability of

- Exactly 4 calls in 2 mins $\Rightarrow \lambda' = \lambda \times 2$ $\quad P_X(4) = \frac{(2\lambda)^4 e^{-2t}}{4!}$
- At least 3 calls in 1 min $\Rightarrow \lambda' = \lambda \times 1 = \lambda$
- 5 calls in 5 mins, given 2 calls in the third mins $\quad P(A|B)$
- 5 calls in 5 mins, among which 2 calls in the third min $\quad P(AB)$

$1 - \sum_{k=0}^{2} P_X(k)$



1 2 3 4 5 min

$\underbrace{\quad}_{2\lambda}$ 2 $\underbrace{\quad}_{2\lambda}$

0 1 2 3

3 2 1 0

$$\Rightarrow \frac{(2\lambda)^0 e^{-2\lambda}}{0!} \cdot \frac{(2\lambda)^3 e^{-2\lambda}}{3!}$$

$\Rightarrow$ Alt.

$\hookrightarrow$ 4 calls in 4λ seg.

$$\Rightarrow \frac{(4\lambda)^3 e^{-4\lambda}}{3!}$$

$$\Rightarrow \frac{(4\lambda)^3 e^{-4\lambda}}{3!} \times \frac{\lambda^2 e^{-\lambda}}{2!}$$

Scaling of PDF

$X = \frac{Y-b}{a}$ ↗ Expand x support

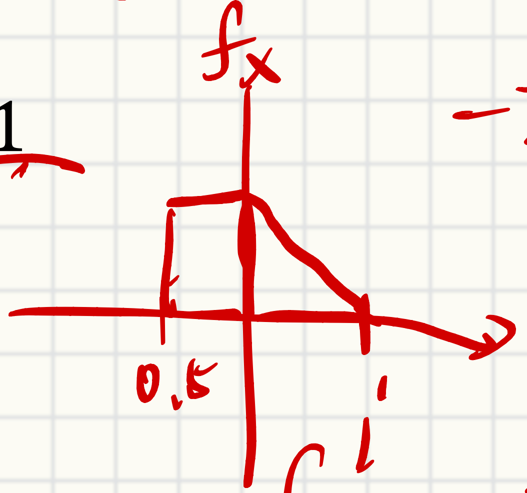
$f_Y(c) = f_X\left(\frac{c-1}{2}\right) \times \frac{1}{2} = \frac{1-\left(\frac{c-1}{2}\right)}{2} = \frac{3-c}{4}$

Let $Y = aX + b$, where X, Y are RV and a, b are constants

- $f_Y(u) = f_X\left(\frac{u-b}{a}\right) \times \boxed{\frac{1}{|a|}}$ → density ↓ by a times

$f_Y(c)\begin{cases} \frac{1}{2} & 0 - 1 \\ \frac{3-c}{4} & 1 - 3 \end{cases}$

$f_X(u) = \begin{cases} 1 & -0.5 \le u \le 0 \\ 1 - u & 0 \le u \le 1 \end{cases}$, let $Y = 2X + 1$



$\sigma_Y^2 = 4\sigma_X^2$

- Solve $E[Y]$ and σ_Y^2

$P\{2X+1 \ge 2\}$

$E[Y] = 2E[X] + 1$

- $P\{Y \ge 2\}$

$= P\{X \ge \frac{1}{2}\} = \int_{\frac{1}{2}}^{1} (1-u)\,du$

$E[X] = \int_{-0.5}^{1} u f_X(u)\,du$ $\int_0^1 u - u^2$

$= -0.25 + 0.16$

Markov and Chebyshev

$= \int_2^3 f_Y(u)\,du.$

$1-\left(\frac{u-1}{2}\right)$ $\quad \frac{u^2}{2}\Big|_0^1 - \frac{u^3}{3}\Big|_0^1$

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Markov

- Check whether Y is none-negative 2.
- Compute $E[Y]$ and σ_X carefully

$Y = (X-\mu_X)^2$

Markov

- $P\{Y \geq c\} \leq \frac{E[Y]}{c}$
- $p_X(k) = \begin{cases} 1/6 & 1 \leq k \leq 6 \\ 0 & else \end{cases}$

$P\{X \geq 4\} \leq \frac{E[X]}{4} = \frac{3.5}{4} = 0.875$

Chebyshev

- $P\{|X - \mu_X| \geq a\sigma_X\} \leq \frac{1}{a^2}$

$P\{(X-3.5) \geq 0.5\} \leq$

$P\{|X-3.5| \geq 0.5\}$

$a\sigma_X$

$\leq \frac{1}{a^2}$

PDF/ CDF

CDF

- Properties – None-decreasing/ output span [0,1]/ right cont.
- $F_X(c) = \int_{-\infty}^{c} f_x(u)du$– Remember the constant
- “Jump” at c ($F_X(c) > \mathrm{F_X(c-)}$) implies $p_X(c) > 0$

PDF

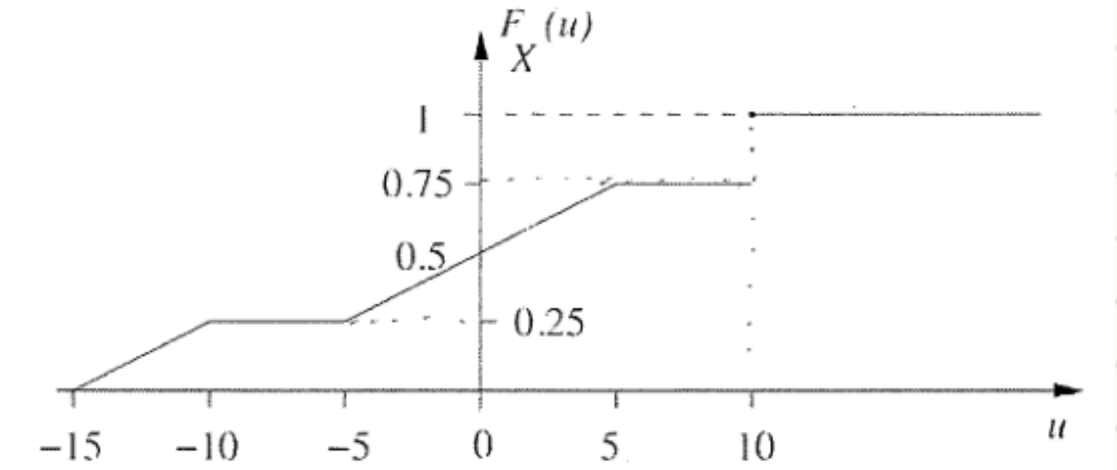
- $f_X = F_X{}'$
- $\int_{-\infty}^{\infty} f_X(u)du = 1$

$$P\{a < X \leq b\} = F_X(b) - F_X(a) = \int_a^b f_X(u)du$$

PDF/ CDF

$$f_X(x) = \begin{cases} c^2 e^{-5x} & x \geq 0 \\ 0 & else \end{cases}$$, find c, $F_X(x)$, $P\{1 \leq X \leq 3\}$

PDF/ CDF



- Solve $P\{X \leq -5\}, P\{X = 10\}, P\{X^2 \leq 4\}, E[X]$

Real-Time QA