Last lecture

Random Variables (RV)

- Variance (Ch 2.2)

Conditional Probability (Ch 2.3)

- Motivation
- Examples

Agenda

Conditional Probability (Ch 2.3)

- Examples
- Solver
- 3 doors problem revisited

Law of Total Probability (Ch 2.10)

- Bayes formula

X = roll D6

$Var(X) =$

① $E[(X-\mu_x)^2] = \frac{1}{6}((-2.5)^2 + (-1.5)^2 \dots + (2.5)^2)$

② $\underline{E[X^2]} - \mu_x^2$

$= \frac{1}{6}(1+4+9+16+25+36) - 3.5^2$

X	$X - \mu_x$	P
1	-2.5	$\frac{1}{6}$
2	-1.5	⋮
3	-0.5	⋮
4	0.5	⋮
5	1.5	
6	2.5	$\frac{1}{6}$

Conditional Probability

Conditional Probability

given it happens

$$P(B|A) = \begin{cases} \dfrac{P(A\cap B)}{P(A)} & \text{If } P(A) > 0 \\ \text{Undefined} & \text{Else} \end{cases}$$

$P(A)=0$

	A	A^c
B	4	26
B^c	1	5

Roll two dice, A = sum is 6; B = numbers are not equal

$(3,3)$

$P(B)$ =? $P(B|A)$ =? $P(B^c|A)$ =?

$(1,1)$

$(2,2)$

$\vdots$

$(6,6)$

$$\frac{36-6}{36} = \frac{5}{6}$$

$A \cap B$

1 5 ✓

2 4 ✓

3 3 ✗

4 2 ✓

1 5 ✓

$$\frac{P(AB)}{P(A)} = \frac{\frac{4}{36}}{\frac{5}{36}} = \frac{4}{5}$$

$$P(B^c|A) = \frac{\frac{1}{36}}{\frac{5}{36}} = \frac{1}{5}$$

Conditional Probability

$$P(B|A) = \frac{P(AB)}{P(A)}$$

$P(A, B) = P(B|A) \times P(A)$ If $P(A) > 0$

In many cases, we might only know some probabilities...

- 3 doors problem A: $x_1 = Car$, if we change...
  - $P(W|A) = 0$
  - $P(W, A) = 0$
  - $P(W, A^c)$

$$= P(W|A^c)\, P(A^c)$$

$$= 1 \times \frac{2}{3} = 67\%$$

Facts of conditional probability

- $P(B|A) \geq 0$
- $P(B|A) + P(B^c|A) = 1$
- $P(\Omega|A) = 1$
- $P(AB) = P(A|B)P(B)$
- $P(ABC) = P(A|BC)P(B|C)P(C)$

A^+

2nd

90

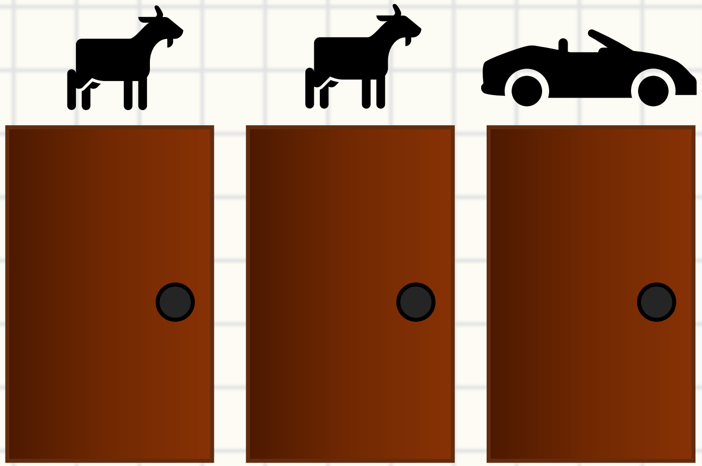
1st

>90

$= P(A|BC)\,P(BC)$

$= P(A|BC)\,P(B|C)\,P(C)$

Examples



- Never change
  - $P(W|X_1 = C) =$
  - $P(W|X_1 = G) =$

- Change
  - $P(W|X_1 = C) =$
  - $P(W|X_1 = G) =$

- What if there are 4 doors… 2 cars and 2 goats?

Slido

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Consider A, B, C each wearing a hat of RED or BLUE

- C is blind
- A and B cannot their own hat, but can see others’
- At least 1 RED hat among A, B, C
- A say “I don’t know the color of my hat” R
- B say “Me neither”
- What is C’s hat color?

A B C

(B , B) X

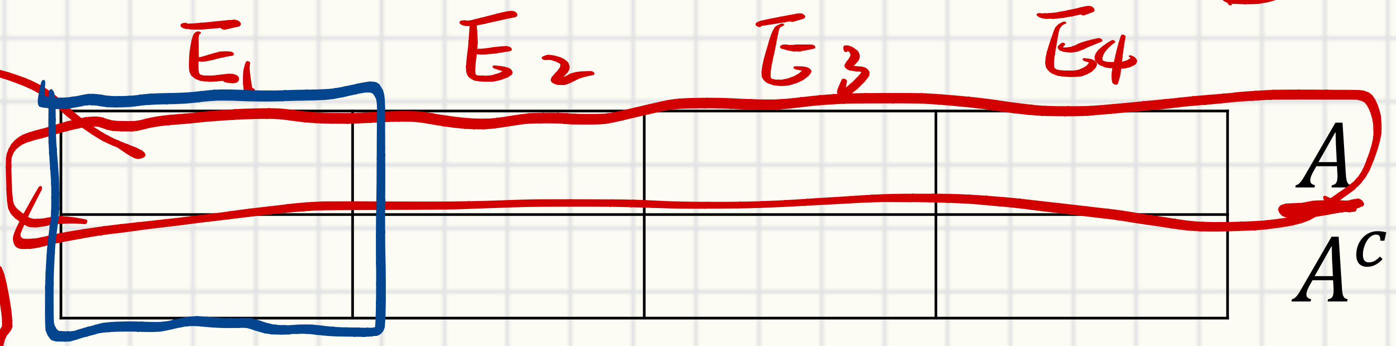
R B → B will know he is R.

Law of total probability (Ch 2.10)

Law of total probability

Partition

$P(A E_1) = P(E_1) \cdot P(A|E_1)$



$P(A)$

- Case-by-case discussion law…
- $P(A)$ is the summed of "Partitioned conditional probability"
- $P(A) = \sum_i P(A|E_i)P(E_i)$

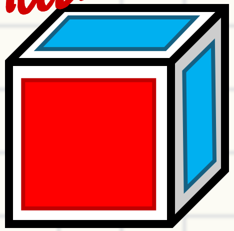
$= \sum_i P(A E_i)$

$P(W \cap X_1 = C) +$
$P(W \cap X_1 = G) = P(W)$

Law of total probability

There are 3 dice A, B, C in the bag

- $A = [R \times 1; B \times 5]$ ⇒ almost blue
- $B = [R \times 2; B \times 4]$
- $C = [R \times 3; B \times 3]$



Draw one die and roll many times

R_1 : First roll RED

- $P(R_1) = P(R_1, A) + P(R_1, B) + P(R_1, C)$
- $P(R_2|R_1)$

$= P(R_1|A)P(A) + \cdots$

$= \frac{1}{6} \times \frac{1}{3} + \frac{2}{6} \times \frac{1}{3} + \frac{3}{6} \times \frac{1}{3} = \frac{1}{3}$

$$P(R_2|R_1) = \frac{P(R_2R_1)}{P(R_1)}$$

modify

$$\underbrace{P(R_1R_2|A)}_{\frac{1}{36}}\,\underbrace{P(A)}_{\frac{1}{3}} + \underbrace{P(R_1R_2|B)}_{\frac{4}{36}}\,\underbrace{P(B)}_{\frac{1}{3}} + \underbrace{P(R_1R_2|C)}_{\frac{9}{36}}\,\underbrace{P(C)}_{\frac{1}{3}}$$

Bayes Formula

Conditional probability + Law of total probability

- How do we get $P(B|A)$ from $P(A|B)$?

- $P(B|A) = \dfrac{P(AB)}{P(A)} = \dfrac{P(A|B)\,P(B)}{P(A)}$

cond.

- $P(E_i|A) = \dfrac{P(A|E_i)\,P(E_i)}{\sum_j P(A|E_j)\,P(E_j)}$

← cond. prob

← total prob

Disease problems

Assume there is a disease A, and the corresponding test T

- What do the followings mean?
- $P(T|A) = 0.9$ $\longleftrightarrow$ $P(T^c|A) = 0.1$
- $P(T|A^c) = 0.05$ $\longleftrightarrow$ $P(T^c|A^c) = 0.95$
- $P(A) = 0.01$

- $P(A|T) =$

Disease problems

According to CDC survey on smoker

- 18% of adults are smokers
- 15% of women are smokers
- Population = 50% men + 50% women

- What fraction of adult smokers are women