Last lecture

Random Variables (RV)

- Probability Mass Function (pmf)
- Mean and Variance (Ch 2.2)

Agenda

Random Variables (RV)

- Mean and Variance (Ch 2.2)

Conditional Probability (Ch 2.3)

- Motivation
- Examples
- Solver

Law of Total Probability (Ch 2.10)

Variance and Standard Deviation

Mean is important… but not complete enough

- Variance $Var(X)$ is how PMF spreads apart from μ_X

- $Var(X) \triangleq E[(X - \underset{\downarrow \atop E[X]}{\mu_x})^2] = E[X^2] - (\underbrace{E[X]}_{\mu_x})^2$

unit^2. e.g. m^2 if $x = ?\ m$.

Variance and Standard Deviation

exam-safe

Standard deviation $\sigma_X \triangleq \sqrt{Var(X)}$; $Var(X) = \sigma_X^2$

μ_X

σ_X

σ_X is of the same unit as X

$Var(X + c) =$ $E[((X+\cancel{c}) - \underbrace{E[X+c]}_{E[X]+\cancel{c}})^2]$ $= E[(X - E[X])^2]$ $= Var(X)$

$Var(aX + \cancel{c}) =$ $E[(aX)^2] - (E[aX])^2$

$2X$ vs $X + X$

$Var(aX) = a^2 Var(X) = E[a^2X^2] - (a\mu_X)^2 = a^2(\underbrace{E[X^2] - \mu_X^2}_{Var X})$

$$Y = X + c \qquad Var(Y) = E[(Y - E[Y])^2]$$

$$E[(X+c)] = E[X] + c$$

$$= E[[(X + \cancel{c}) - (E[X] + \cancel{c})]^2]$$

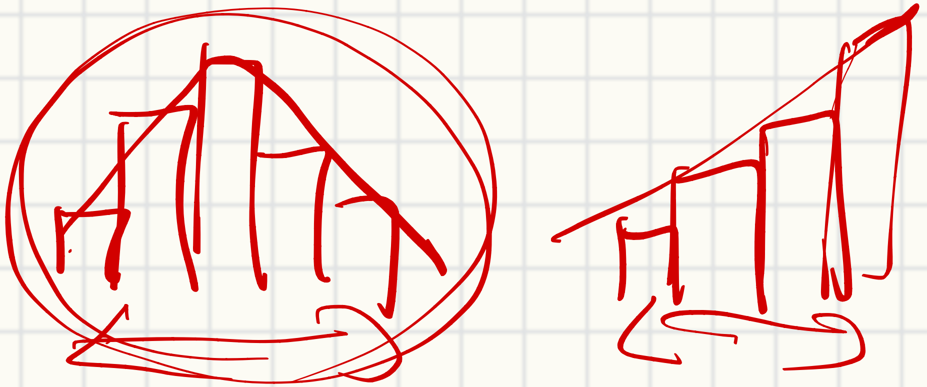
$$Var(2X) = 2^2 \, Var(X) = 4 \, Var(X)$$

$$Var(X_1 + X_2) = E[(X_1 + X_2 - \mu_{X_1} - \mu_{X_2})^2]$$

$$= E[[(X_1 - \mu_{X_1}) + (X_2 - \mu_{X_2})]^2]$$

$$= Var(X_1) + Var(X_2) = 2 \, Var(X)$$

Standardized RV



For any RV X

- $\frac{X-\mu_X}{\sigma_X}$ is a standerized RV – Mean 0, variance 1

$$E\left[\frac{X-\mu_x}{\sigma_x}\right] = \frac{E[X]-\mu_x}{\sigma_x} = \frac{\mu_x-\mu_x}{\sigma_x} = 0$$

$$Var\left[\frac{X}{\sigma_x} - \cancel{\frac{\mu_x}{\sigma_x}}\right] = \left(\frac{1}{\sigma_x}\right)^2 Var\,X = \frac{\sigma_x^2}{\sigma_x^2} = 1$$

<u>Mean</u> μ_X $E[f(X)] \neq f(E[X])$

$$\begin{cases} E[aX+c] = aE[X]+c. \\ Var(aX+c) = a^2\, Var(X) \end{cases}$$

$Var(X)$

$\Rightarrow$ $Var(X+Y) = Var\,X + Var\,Y$

if X, Y are independent.

Conditional Probability

Motivation

The probability of B happens given A happens

- $P($ pair of socks are same color $)$ given $S_1 = B$
- $P(I\ win\ Texas\ Hold'em)$ given $X = Ace + Ace$
- $P(I\ pass\ 313)$ given I skip HW1…

Why do we need conditional probability?

- Analyze the relationship between two events
- Find the optimal solution to make an event probable

Examples

$P(win \mid x_1)$ 50 ① 33%

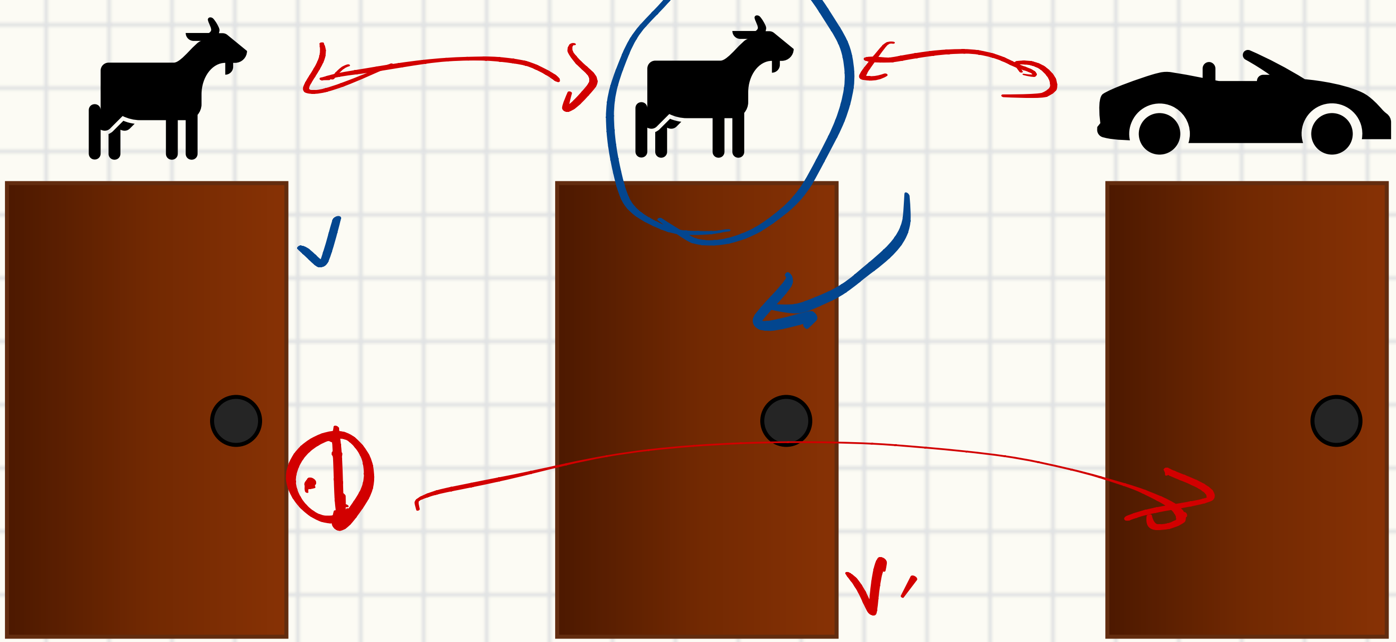
50 ② 50%. 67%

3 doors problem

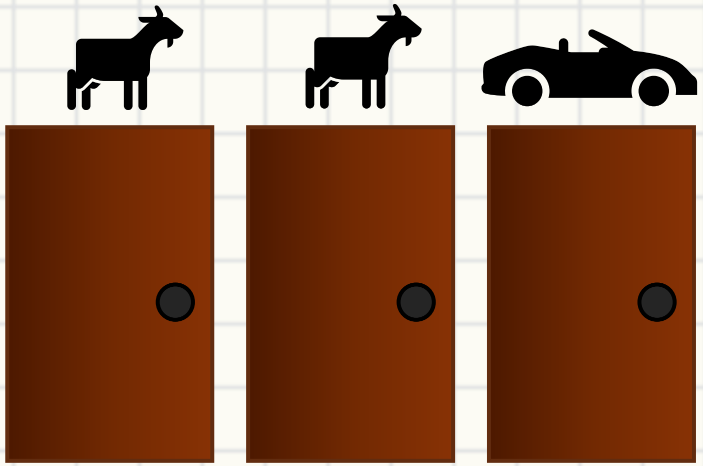
- 3 closed doors – 1 leads to a car, the others lead to goats
- After you choose one, the host will open a "Fail" door
- Should you change the door?

A

B



Examples



$P(X_1 = C_1)$

- Never change
  - $P(W|X_1 = C) =$ 100%
  - $P(W|X_1 = G) =$ 0%

$$\begin{cases} P(W, X_1 = C) = 100\% \times \frac{1}{3} = 33\% \\ P(W, X_1 = G) = 0\% \end{cases}$$

- Change
  - $P(W|X_1 = C) =$ 0%
  - $P(W|X_1 = G) =$ 100%

$$\begin{cases} P(W, X_1 = C) = 0\% \\ P(W, X_1 = G) = 100\% \times \frac{2}{3} \end{cases}$$

$= 67\%$

- What if there are 4 doors… 2 cars and 2 goats?

2C 2G

No changing. 50%

Change. $P(W \mid X_1 = G) = 100\%$

$P(W \mid X_1 = C) = 50\%$

$P(X_1 = C)$

$P(W, X_1 = G) = 1 \times \frac{1}{2} = \frac{1}{2}$

$75\% + P(W, X_1 = C) = 50\% \times \frac{1}{2} = 25\%$

Conditional Probability

$$P(B|A) = \begin{cases} \dfrac{P(AB)}{P(A)} & \text{if } P(A) > 0 \\ \text{undefined} & \text{if } P(A) = 0 \end{cases}$$

Roll two dice, A = sum is 6; B = numbers are not equal

$P(B) =?$ $P(B|A) =?$ $P(B^c|A) =?$