Last lecture

Minimum mean square error estimation (Ch 4.9)

- Recap
  - Constant estimators
  - Unconstrained estimators
  - Linear estimators
- Examples

Agenda

Joint Gaussian Distribution (Ch 4.11)

- Motivation
- Facts
- Examples

Joint Gaussian distribution

Motivation

Describe the joint distribution of (Height, Weight)=(X, Y) of the class

- Metrics: $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y}$
- Is $aX + bY$ Gaussian? NOT necessarily, but if TRUE

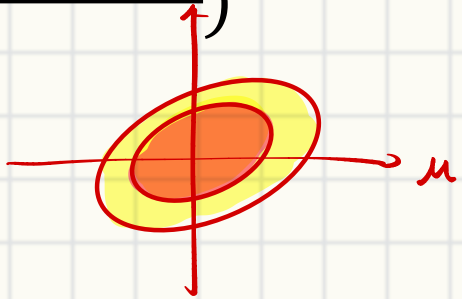
Def. 4.11.1 RV X and Y are said to be **jointly Gaussian** if every linear combination $aX + bY$ is a Gaussian random variable

- $$f_{X,Y}(u,v) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left(-\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2(1-\rho^2)}\right)$$

$Z = aX + bY$,

(Z, X) / (Z, Y) jointly G.

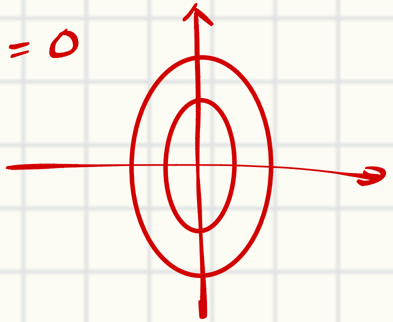
↳ 2D Gaussian if X, Y are independent



Justifying a bivariate normal

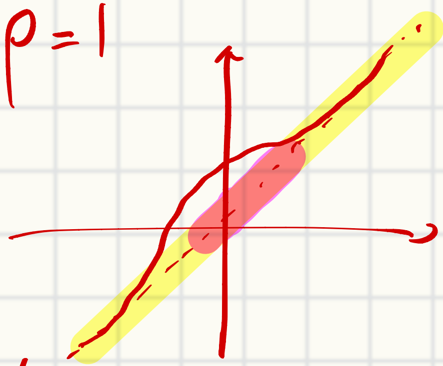
(jointly Gaussian)

$\rho = 0$



A bivariate normal $f_{X,Y}(u,v) = C \times exp\big(-P(u,v)\big)$

$\rho = 1$



- $P(u,v) = au^2 + buv + cv^2 + du + ev + f$
- $P(u,v) \to \infty$ as $|u| + |v| \to \infty$

$\Rightarrow \iint f_{XY}\, du\, dv = 1$

- $a, c > 0,\ b^2 - 4ac < 0$

↳ $-1 \le \rho \le 1$

$Y = 2X$

$\sigma_Y = 2\sigma_X$

Standard 2-d Normal to Bivariate Normal

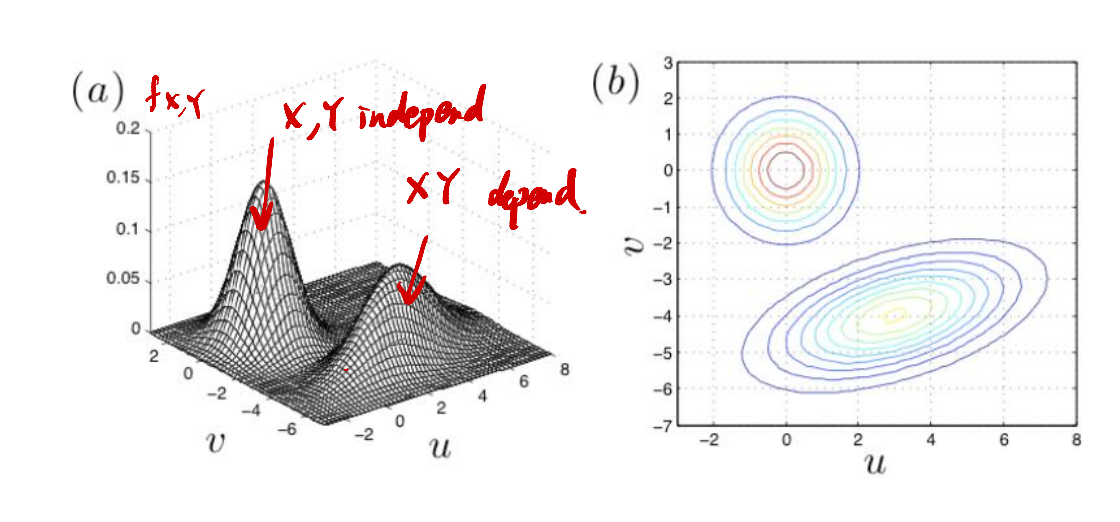
Let W, Z be independent standard normal

- $f_{W,Z}(\alpha,\beta) = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}\frac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi}} = \frac{e^{-\frac{\alpha^2+\beta^2}{2}}}{2\pi}$

  $f_W(\alpha)\quad f_Z(\beta)$

- $\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} W \\ Z \end{pmatrix} + \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$

- $A = \begin{bmatrix} \overset{a_{11}}{\sqrt{\frac{\sigma_X^2(1+\rho)}{2}}} & \overset{a_{12}}{-\sqrt{\frac{\sigma_X^2(1-\rho)}{2}}} \\ \overset{a_{21}}{\sqrt{\frac{\sigma_Y^2(1+\rho)}{2}}} & \overset{a_{22}}{\sqrt{\frac{\sigma_Y^2(1-\rho)}{2}}} \end{bmatrix}$



$$Var(X) = \sigma_X^2 = Cov(\underbrace{a_{11}W + a_{12}Z}_{X},\ a_{11}W + a_{12}Z)$$

$$= a_{11}^2 + a_{12}^2 = \frac{\sigma_X^2(1+\rho)}{2} + \frac{\sigma_X^2(1-\rho)}{2} = \sigma_X^2$$

$$\rho_{XY} = \frac{Cov(\overbrace{a_{11}W + a_{12}Z}^{X},\ \overbrace{a_{21}W + a_{22}Z}^{Y})}{\sigma_X \sigma_Y} = \frac{a_{11}a_{21} + a_{12}a_{22}}{\sigma_X \sigma_Y}$$

$$= \rho$$

Facts

$f_{XY} = \underbrace{f_X(u)}_{\sim N(\mu_X, \sigma_X^2)} f_{Y|X}(v|u)$

$\sim N(\rho\mu, \sigma_Y^2(1-\rho^2)) N(\mu_X, \sigma_X^2)$

If X and Y forms the bivariate normal with $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$

- $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$

$f_{XY}(u,v) = \left[\frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(u-\mu_X)^2}{2\sigma_X^2}\right)\right]$

- $aX + bY$ is a Gaussian for any $a, b \in \mathbb{R}$

$\left[\frac{1}{\sqrt{2\pi\sigma_Y^2(1-\rho^2)}} \exp\left(-\frac{(v-\rho u)^2}{2\sigma_Y^2(1-\rho^2)}\right)\right]$

- $\rho = \rho_{X,Y}$, X and Y are independent iif $\rho = 0$

$f_{Y|X} \sim N(\rho u, \sigma_Y^2(1-\rho^2))$

- $E[Y|X] = \hat{E}[Y|X]$ $\quad g^*(X) = aX+b$

- $(Y|X = u) \sim N(\hat{E}[Y|X = u], \sigma_e^2)$

Example

Let X and Y be jointly Gaussian with mean $(0,0)$. $\sigma_X^2 = 5$, $\sigma_Y^2 = 2$ and $Cov(X, Y) = -1$. Find $P\{X + 2Y \geq 1\}$ in terms of Φ function

- $Z = X + 2Y$ is Gaussian $aX + bY$ $a = 1$ $b = 2$
- $\mu_Z = \mu_X + 2\mu_Y = 0 + 0 = 0$
- $Var(Z) =$

$Cov(X+2Y, X+2Y) = Var(X) + 4\,Cov(X,Y) + 4\,Var(Y)$

$= 5 + (-4) + 4 \times 2 = 9$

$Z \sim N(0, 9)$ $P\left\{\frac{Z}{3} \geq \frac{1}{3}\right\} = Q\left(\frac{1}{3}\right) = 1 - \Phi\left(\frac{1}{3}\right)$

Example

$Y|X \sim N$

$Y^2|X \not\sim N$

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X,Y) = \rho$. Find $E[Y^2|X]$ $\Rightarrow$

- $\hat{E}[Y|X] = \cancel{\mu_Y}^{0} + \frac{Cov(X,Y)}{\cancel{Var(X)}_{1}}(X - \cancel{\mu_X}^{0}) = \rho X$
- For any RV, $Var(Z) = E[Z^2] - (E[Z])^2$

$Z = Y|X$

$$Var(Y|X) = E[Y^2|X] - (E[Y|X])^2$$

$\hat{E}[Y|X] = E[Y|X]$, $(E[Y|X])^2 = (\rho X)^2$

MSE of $\hat{E}[Y|X] = \sigma_Y^2(1-\rho^2)$

$\hookrightarrow (1-\rho^2) + (\rho u)^2$

Example

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X, Y) = 0.5$. Find

- $Var(3X - 2Y)$

- $P\{(3X - 2Y)^2 \leq 28\}$ in terms of Φ

- $E[Y|X = 3]$

Quick questions

Which plot shape best represents the contour lines of a bivariate Gaussian?

A. Circles B. Ellipses C. Triangles D. Rectangles

If two Gaussian random variables X and Y are independent, what must be true?

ρ

A. $\sigma_{XY} = 0$ B. Their means must be equal C. Their variances must be equal D. Nothing specific

Suppose (X,Y) is jointly Gaussian. If ρ=0, what is the resulting joint density shape?

A. A tilted ellipse B. A vertical ellipse C. A circle D. A horizontal straight line