Last lecture

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)
- Examples for
  - General
  - X is Gaussian
  - Case by case
  - g is cosine/ tangent

Agenda

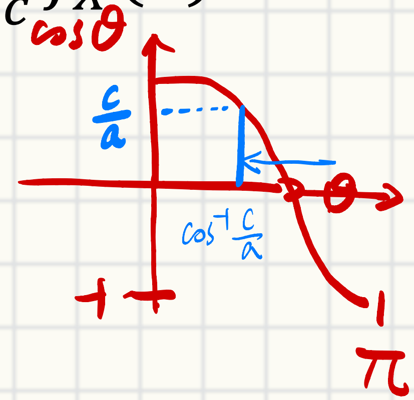
Functions of a random variable (Ch 3.8)

- Examples for
  - g is cosine/ tangent
  - g is strictly increasing

Generating Customized RV X from F_X (Ch 3.8.2)

Examples – Cosine

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$



Let $B = a\cos\Theta$, where $\Theta \sim Uni(-\pi,\pi)$, find f_B

- Useful in DSP
- $F_B(c) = P\{a\cos\theta \le c\} = P\{\theta \ge \cos^{-1}(\frac{c}{a})\}$

  $= 1 - \frac{\cos^{-1}(\frac{c}{a})}{\pi}$ $\quad 0<\theta<\pi.$

- $f_B(c) = F_B{}'(c) = \frac{1}{\pi\sqrt{a^2-c^2}}$
  - Hint $\frac{d(\cos^{-1}x)}{dx} = -(1-x^2)^{-\frac{1}{2}}$

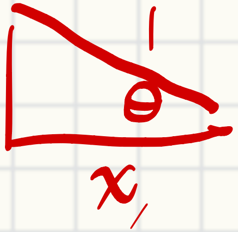
$d\cos^{-1}(\frac{c}{a}) = -\frac{1}{\sqrt{1-\frac{c^2}{a^2}}} \times \frac{1}{a}$

$\cos\theta = x,$

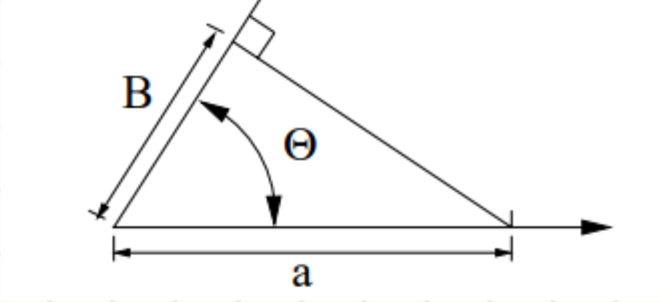
$\frac{d\theta}{dx}$

$-\sin\theta \frac{d\theta}{dx} = 1$

$\frac{1}{\sqrt{1-x^2}}$



$\frac{d\theta}{dx} = \frac{1}{-\sin\theta} = \frac{1}{-\sqrt{1-x^2}}$



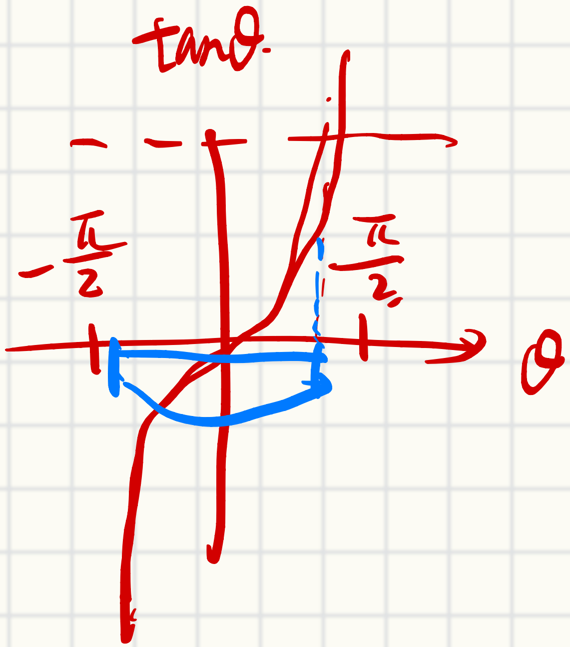
Examples – Tangent

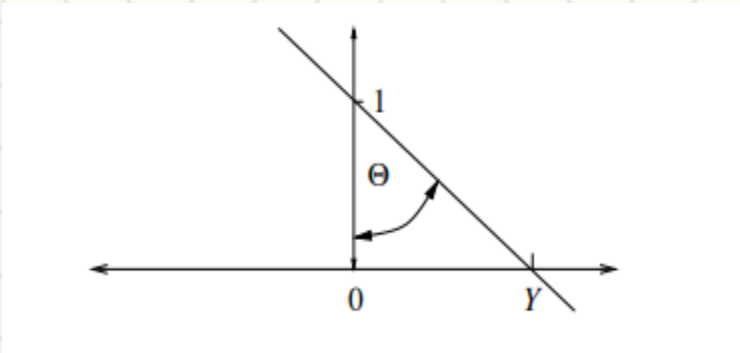
1. Find support and continuity
2. $F_Y(c) = \int_{x: f(x) \le c} f_X(x)dx$
3. $f_Y = F_Y'$

Let $Y = \tan\Theta$, where $\Theta \sim Uni\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find f_Y

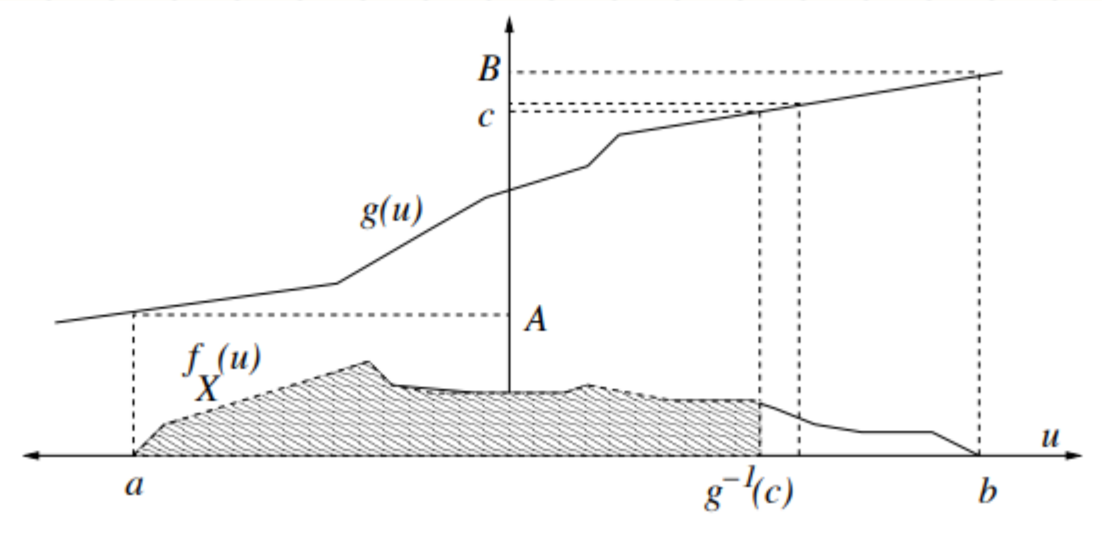
- $F_Y(c) = P\{\Theta \le \tan^{-1}(c)\} = \dfrac{\tan^{-1}(c) + \frac{\pi}{2}}{\pi}$

- $f_Y(c) = F_Y'(c) = \dfrac{1}{\pi\sqrt{1+c^2}}$





Increasing g function



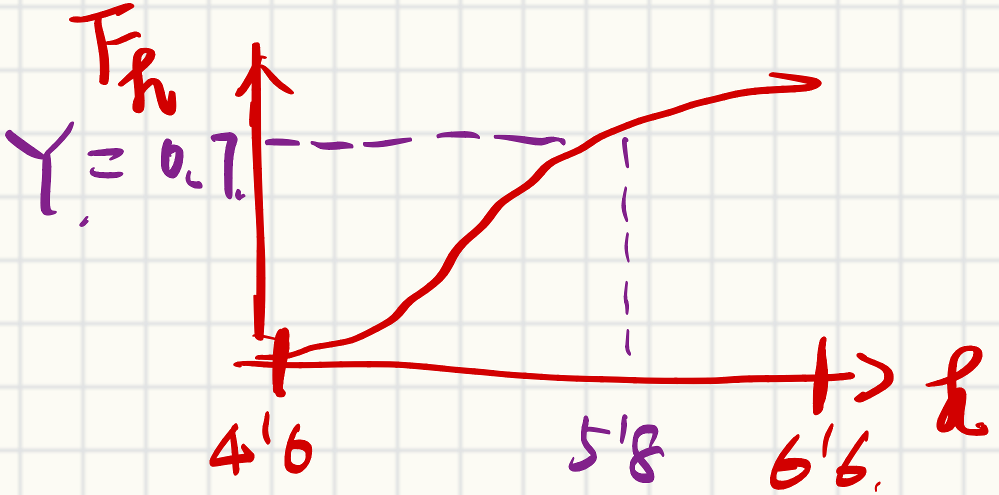
- X has support $[a, b]$
- $Y = g(X)$
- g is strictly increasing
- Y has support $[g(a), g(b)] = [A, B]$
- Find $F_Y(c)$ where $A \leq c \leq B$

$$F_Y(c) = P\{X \leq g^{-1}(c)\} = F_X(g^{-1}(c))$$

- $f_Y(c) =$ $F_X'(g^{-1}(c)) = f_X(g^{-1}(c)) \dfrac{1}{g'(g^{-1}(c))}$

$g(g^{-1}(c)) = c \hookrightarrow \dfrac{d}{dc}$ $\quad g'(g^{-1}(c)) \dfrac{d\, g^{-1}(c)}{dc} = 1$

Creating uniform distribution



- Let $Y = F_X(X)$
- What is F_Y?

$Y \in [0, 1]$

$$F_Y(c) = P\{F_X(X) \le c\} = c.$$

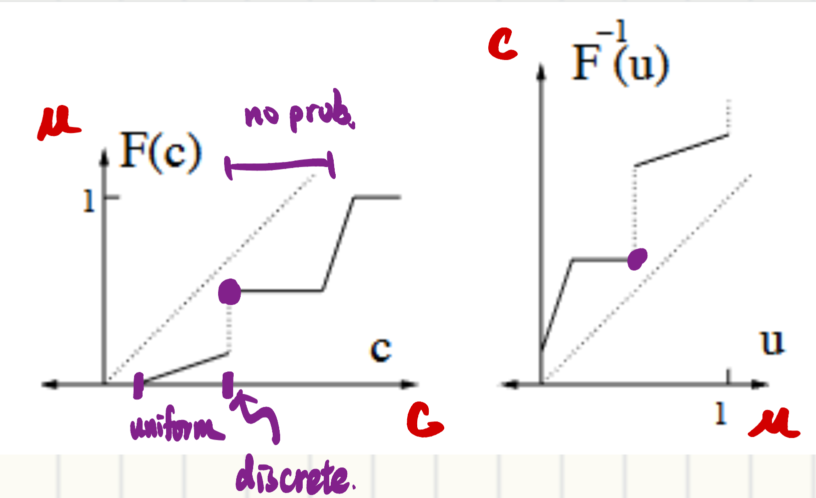
$$\Rightarrow Y = \text{Uniform}([0, 1])$$

Generating a customized RV

If we want to generate (sample) $X = g(U \sim Uni([0,1]))$ to fit F_X

- $U = F_X(X)$
- $F_X^{-1}(U) = X$
- $g = F_X^{-1}$, but F_X^{-1} may not be a function
    - $F_X(c_1) = F_X(c_2)$
- For any F, define

$$F^{-1}(u) = \min\{c: F(c) \geq u\}$$



$F^{-1}(u_0) \leq c_0$ if and only if $u_0 \leq F(c_0)$

- $F_X(c) = P\{F^{-1}(U) \leq c\} = P\{U \leq F(c)\} = F(c)$