Last lecture

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Examples

ML estimation for continuous RVs (Ch 3.7)

- Definition
- Examples

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)

Agenda

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)
- Examples for
  - General
  - X is Gaussian
  - Case by case
  - g is cosine/ tangent
  - g is strictly increasing

Find CDF/ PDF of g(X)

Motivation – I know X follows some distribution

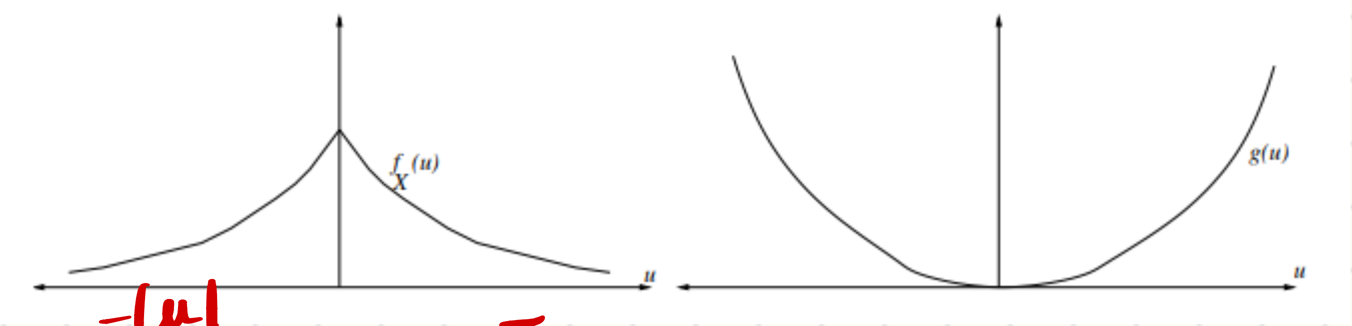
- but what about $Y = g(X)$?

1. Scope the problem - Find support of X and Y, are they continuous or discrete?
2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
   - If Y is discrete, normally we can find pmf $p_Y(c)$
3. Get $f_Y = F_Y{}'$

Examples - General

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

RV X follows $f_X(u) = \dfrac{e^{-|u|}}{2}$ for $u \in \underline{R}$. $Y = X^2$. Find f_Y, μ_Y and σ_Y^2



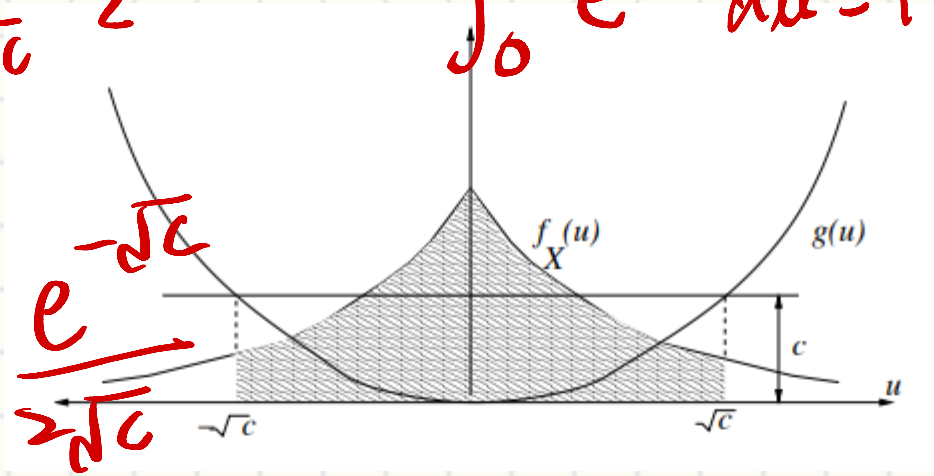
1. $F_Y(c) = \begin{cases} 0 & c \leq 0 \\ ?? & c > 0 \end{cases}$

   $\parallel$

2. $P\{X^2 \leq c\} =$

$$\int_{-\sqrt{c}}^{\sqrt{c}} f_X(u)du = \int_{-\sqrt{c}}^{\sqrt{c}} \frac{e^{-|u|}}{2} du = \int_0^{\sqrt{c}} e^{-u} du = 1 - e^{-\sqrt{c}}$$

$$= 1 - e^{-c^{\frac{1}{2}}}$$



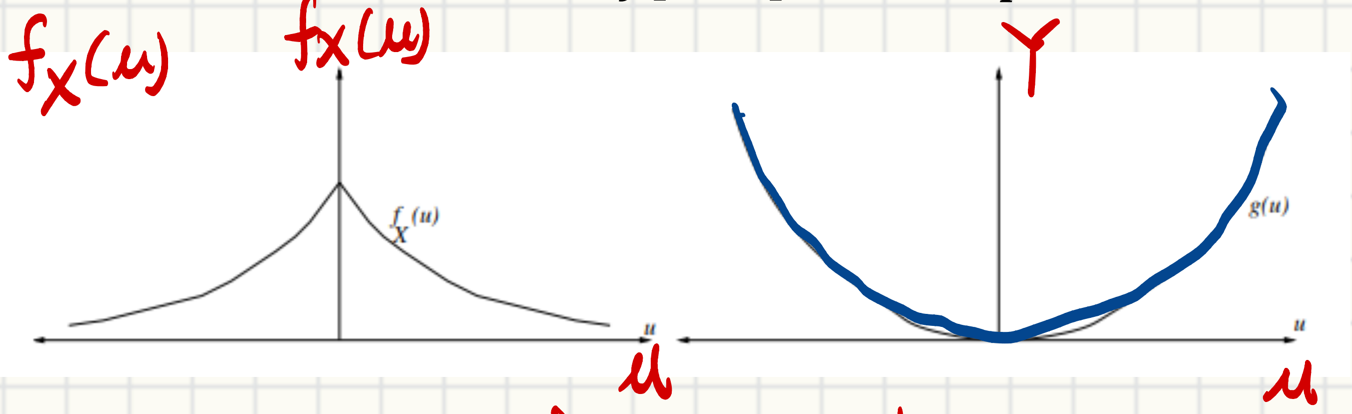
3. $f_Y(c) = F_Y'(c)$

$$= e^{-c^{\frac{1}{2}}} \cdot \frac{1}{2} \cdot c^{-\frac{1}{2}} = \frac{e^{-\sqrt{c}}}{2\sqrt{c}}$$

Examples - General

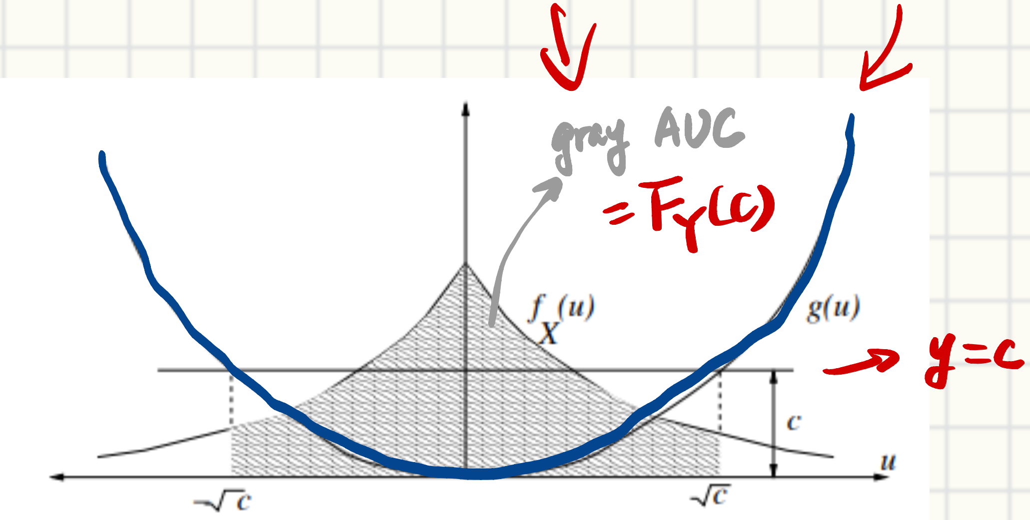
1. Find support and continuity
2. $F_Y(c) = \int_{x: f(x) \le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

RV X follows $f_X(u) = \dfrac{e^{-|u|}}{2}$ for $u \in R$. $Y = X^2$. Find f_Y, μ_Y and σ_Y^2



1. $F_Y =$

2. $P\{X^2 \le c\} =$



3. $f_Y(c) =$

Examples - Gaussian

1. Find support and continuity
2. $F_Y(c) = \int_{x: f(x) \le c} f_X(x)\,dx$
3. $f_Y = F_Y{}'$

Let $Y = X^2$, where $X \sim N(\mu = 2, \sigma^2 = 3)$, find f_Y in Φ

- $F_Y(c) = P\{-\sqrt{c} \le X \le \sqrt{c}\} = P\left\{ \overset{k'}{\frac{-\sqrt{c}-2}{\sqrt{3}}} \le \frac{X-2}{\sqrt{3}} \le \overset{k}{\frac{\sqrt{c}-2}{\sqrt{3}}} \right\}$

$$= \begin{cases} \Phi\left(\frac{\sqrt{c}-2}{\sqrt{3}}\right) - \Phi\left(\frac{-\sqrt{c}-2}{\sqrt{3}}\right) & \text{if } c > 0 \\ 0 & \text{else} \end{cases}$$

- $f_Y(c) = F_Y{}'(c)$

$$\Phi'(s) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-s^2}{2}\right)$$

$$f_Y = \frac{1}{\sqrt{24\pi c}} \left\{ \exp\left(\frac{-(\sqrt{c}-2)^2}{6}\right) + \exp\left(\frac{-(\sqrt{c}+2)^2}{6}\right) \right\}$$

Examples – Case by case

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $Y = (X-1)^2$, where $X \sim Uni(0,3)$, find f_Y

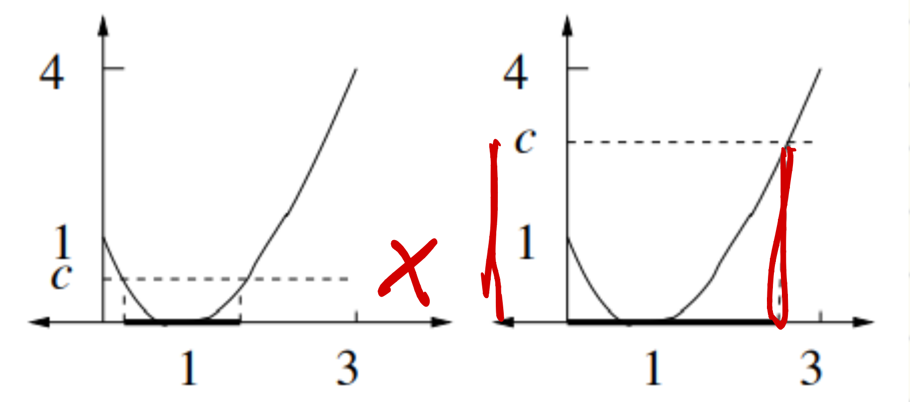
$$F_Y(c) = P\{1-\sqrt{c} \le X \le 1+\sqrt{c}\} = \int_{1-\sqrt{c}}^{1+\sqrt{c}} \frac{1}{3} du = \frac{2\sqrt{c}}{3}$$

- $F_Y(c)$
  - $0 \le c \le 1$
  - $1 \le c \le 4 = (3-1)^2$

$$P\{0 \le X \le 1+\sqrt{c}\} = \frac{1+\sqrt{c}}{3}$$

- $f_Y(c) = F_Y{}'(c)$

Y



$$F_Y(c) = \begin{cases} 0 & c < 0 \\ \frac{2\sqrt{c}}{3} & 0 \le c < 1 \\ \frac{1+\sqrt{c}}{3} & 1 \le c < 4 \\ 1 & c \ge 4 \end{cases}$$

$$f_Y(c) \begin{cases} \frac{1}{3\sqrt{c}} & 0 \le c < 1 \\ \frac{1}{6\sqrt{c}} & 1 \le c < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\int g(u) p_X(u)\, du$$

$$\mu_Y = E[(X-1)^2] = \int_0^3 (u-1)^2 \frac{1}{3}\, du = 1.$$

Midterm Review Poll

- Halloween (Oct.31) class time
- Vote for the contents!
- Optional questions on general classroom feedback

#3976074

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Examples – Cosine

Let $B = acos\Theta$, where $\Theta \sim Uni(-\pi,\pi)$, find f_B

- Useful in DSP
- $F_B(c)$

- $f_B(c) = F_B{}'(c)$
  - Hint $\frac{d(\cos^{-1} x)}{dx} = -(1-x)^{\frac{1}{2}}$