Last lecture

Gaussian (normal) Distribution (Ch 3.6.2)

- Example

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Definition
- CDF Approximation
- Examples

Agenda

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Examples

ML estimation for continuous RVs (Ch 3.7)

- Definition
- Examples

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)

Example

$P\{X<K\} \approx 0.99$

$X \sim Bin(n = 1000, p = 0.5)$, Using Gaussian approximation, find K s.t. $P\{X \geq K\} \approx 0.01 = \mathrm{Q}(2.325)$

$\tilde{X} = 15.8Z + 500$

$Z = \dfrac{\tilde{X} - 500}{15.8}$

- $\mu_X = 500$ $,\sigma_X = \sqrt{np(1-p)} = \sqrt{250} \approx 15.8$
- $P\{X \geq K\} =$
- What if $n = 1000000$?

$$P\left\{\frac{X-500}{15.8} \geq \frac{K-500}{15.8}\right\} \approx P\left\{Z \geq \frac{K-500-0.5}{15.8}\right\} \approx 0.01$$

$= Q(2.325)$

$\dfrac{K-500-0.5}{15.8} = 2.325 \qquad K = 537.26$

$$\mu_x = 500000 \qquad \sigma_x = \sqrt{1M \times 0.5 \times 0.5} = 500$$

$$\frac{K - 500000 - 0.5}{500} = 2.325 \qquad K = 501163$$

Example

$$\frac{X-np}{n} \times \frac{\sqrt{n}}{\sqrt{p(1-p)}}$$

Side note $P\{|\hat{p}-p| \le \frac{a}{2\sqrt{n}}\} \ge 1-\frac{1}{a^2}$

We want to estimate p with $\hat{p} = \frac{X}{n}$

c

- Find $P\{|\hat{p}-p| < \delta\}$ in terms of n, p, δ, and Φ

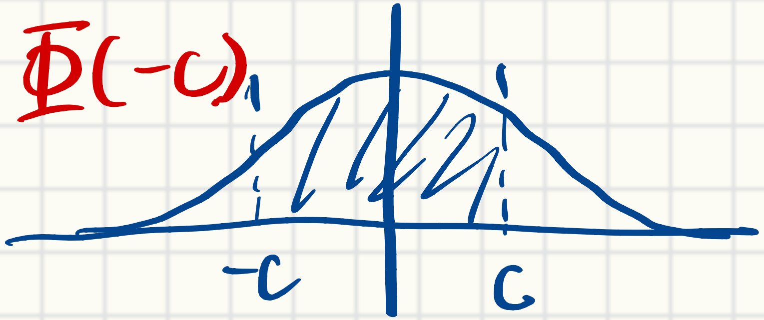
$$P\left\{\left|\frac{X}{n}-p\right| < \delta\right\} = P\left\{\left|\frac{X-np}{\sqrt{np(1-p)}}\right| \le \delta\sqrt{\frac{n}{p(1-p)}}\right\} = P\{|Z| \le c\}$$

- Find δ w/ 99% confidence if $p = 0.5$, $n = 1000$. Given that $\Phi(2.58) \approx 0.995$

- What if $p = 0.1$?

$$P\left\{Z \le \delta\sqrt{\frac{n}{p(1-p)}}\right\} - P\left\{Z \le -\delta\sqrt{\frac{n}{p(1-p)}}\right\}$$

$$= \Phi(c) - \Phi(-c)$$



Ans. ① $\Phi\left(\delta\sqrt{\frac{n}{p(1-p)}}\right) - \Phi\left(-\delta\sqrt{\frac{n}{p(1-p)}}\right) = 2\Phi\left(\delta\sqrt{\frac{n}{p(1-p)}}\right) - 1$

$2\Phi(c) - 1 = 0.99$. $\Phi(c) = 0.995$

② $c = \delta\sqrt{\frac{1000}{0.25}} = 2.58$ $\delta = 0.04$.

③ $c = \delta\sqrt{\frac{1000}{0.1\times 0.9}} = 2.58$ $\delta = 0.025$

Midterm Related

Nov. 3rd 7 PM

HWK 5 — HWK 8 → Gaussian

X cover CLT &

Gaussian Est.

ML estimation

Definition

Recall for discrete RV X, given observation u, ML is to find θ maximizing $p_\theta(u)$

pmf

- But for continuous RV, $p_\theta(u) = 0$

pdf

- Instead, ML maximize $f_\theta(u)$ because
  - $f_\theta(u) \approx \frac{1}{\epsilon} P\{u - \frac{\epsilon}{2} < X < u + \frac{\epsilon}{2}\}$
  - $\hat{\theta}_{ML}(u) \triangleq argmax_\theta \; f_\theta(u)$

Example

$T \sim Exp(\lambda)$ where λ is unknown. We want to estimate λ with observation t

- $f_T(t) =$ $\lambda e^{-\lambda t}$
- Extrema happens at $\frac{df_T(t)}{d\lambda} = 0$

$$e^{-\lambda t} + -(\lambda t) e^{-\lambda t} = 0$$

$$(1-\lambda t) e^{-\lambda t} = 0 \qquad \lambda = \frac{1}{t}$$

Example

$X \sim Uni([0, b])$ where b is unknown. We want to estimate b with observation u

- $f_X(u) = \begin{cases} 0 & \text{if } b < u \\ \frac{1}{b} & \text{if } b \geq u \end{cases}$

- Extrema happens at

$$b = u \Rightarrow f_X(u) = \frac{1}{u}$$

Functions of a random variable

Find CDF/ PDF of g(X)

Motivation – I know X follows some distribution

- but what about $Y = g(X)$? f_Y , μ_Y, σ_Y^2

1. Scope the problem - Find support $\{c: f_X(c) > 0\}$ of X and Y, are they continuous or discrete?
2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
   - If Y is discrete, normally we can find pmf $p_Y(c)$
3. Get $f_Y = F_Y'$

Examples

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

RV X follows $f_X(u) = \frac{e^{-|u|}}{2}$ for $u \in R$. $Y = X^2$. Find f_Y, μ_Y and σ_Y^2

1. $F_Y =$

2. $P\{X \leq c\} =$

3. $f_Y(c) =$