Last lecture

Gaussian (normal) Distribution (Ch 3.6.2)

- Motivation and Definition
- Examples

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Definition
- CDF Approximation
- Examples

Agenda

Gaussian (normal) Distribution (Ch 3.6.2)

- Example

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

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ML estimation for continuous RVs (Ch 3.7)

- Definition
- Examples

Examples

Suppose $\mu_X = 10$ and $\sigma_X^2 = 3$. Compute $P\{X < 10 - \sqrt{3}\}$ if

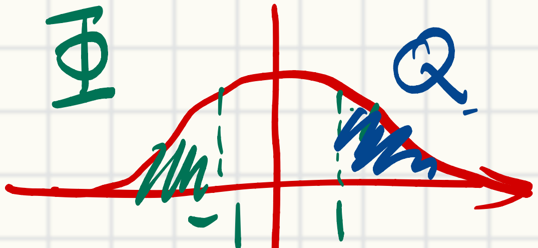
- X is a Gaussian RV in terms of Q
- X is a uniform RV

(Hint: $10 - \sqrt{3} \approx 8.27$)

↳ 1. standardized $Z = \dfrac{X - \mu_X}{\sigma_X}$.

$$Z = \frac{X - 10}{\sqrt{3}}.$$

$$P\{X < 10 - \sqrt{3}\} = P\left\{\frac{X-10}{\sqrt{3}} < \frac{(10-\sqrt{3}) - 10}{\sqrt{3}}\right\}$$



$$= P\{Z < -1\} = \Phi(-1) = Q(1)$$

X is in segment $[\underline{10-k,\ 10+k}]$

$2k$ length.

$Y \sim Uni([0,1])$ $\sigma_Y^2 = \frac{1}{12}$

$X = 2kY + C.$

$\sigma_X^2 = (2k)^2 \sigma_Y^2 = \frac{4k^2}{12} = 3.$

$k^2 = 9$ $k = 3.$

$X \in [7, 13]$ $P\{X < 10 - \sqrt{3}\}$

pdf

$= P\{X < 8.27\} = \frac{8.27 - 7}{13 - 7}$

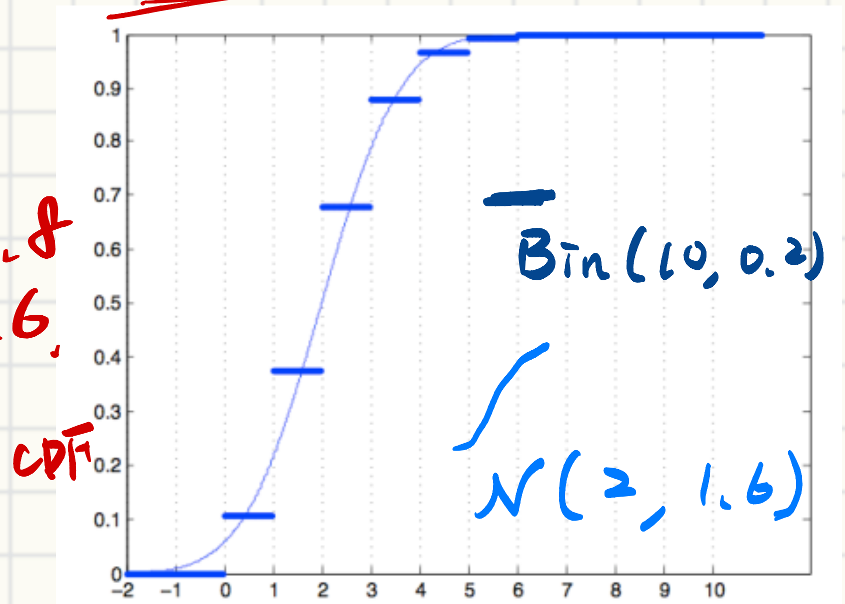
≈ 0.21

Central Limit Theorem and Gaussian Approximation

Central Limit Theorem (CLT)

If many independent random variables are added together, and if each of them is small in magnitude compared to the sum, then the sum X has an approximately Gaussian distribution $\tilde{X}$.

- $P\{X \leq v\} \approx P\{\tilde{X} \leq v\}$
- E.g., $X \sim Binomial(n, p)$ when np and $n(1-p)$ are not small
  - $(n, p) = (10, 0.2)$
  - $\mu_X = np = 10 \times 0.2 = 2$
  - $\sigma_X^2 = np(1-p) = 10 \times 0.2 \times 0.8$
    $= 1.6$
  - What if np is small?



Gaussian Approximation

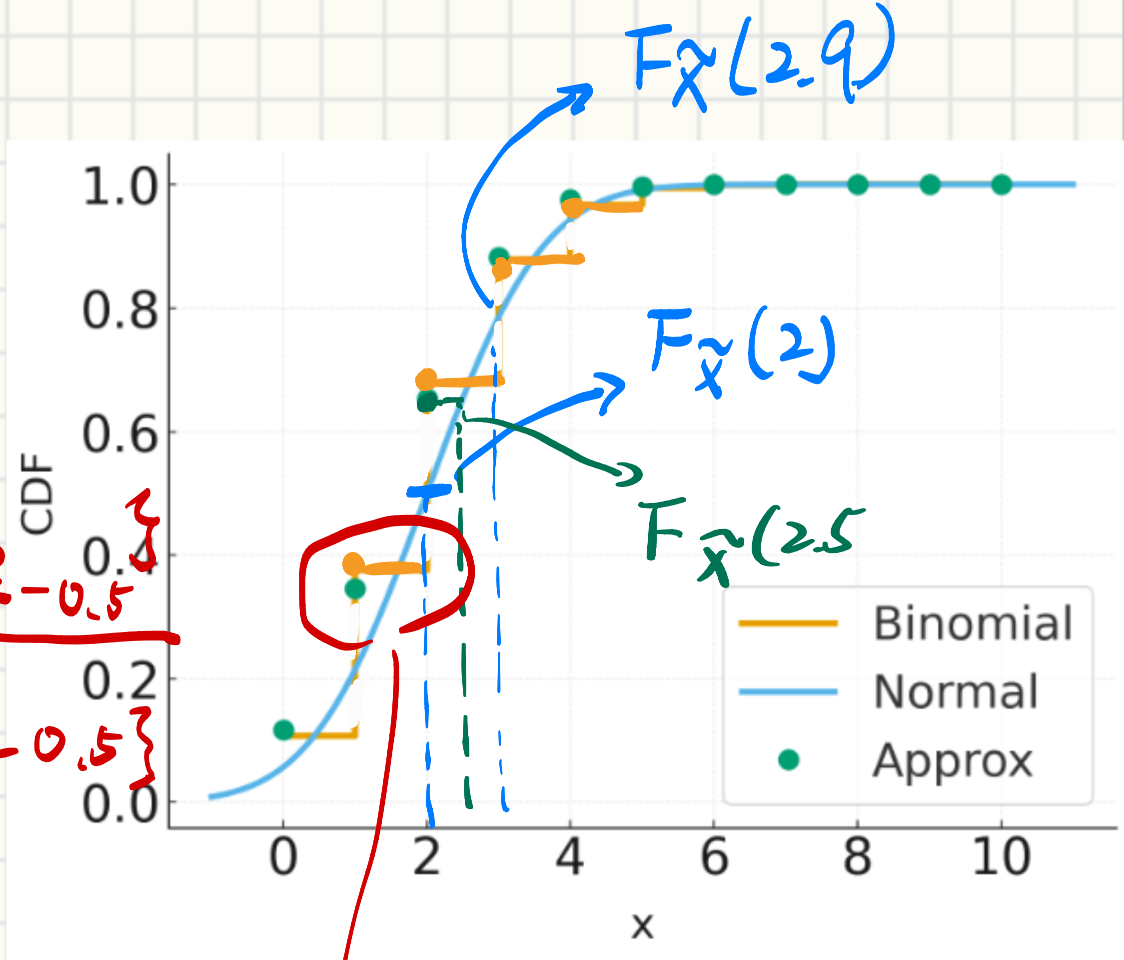
Approximate $X \sim Bin(10, 0.2)$ with $\tilde{X} \sim N(2, 1.6)$

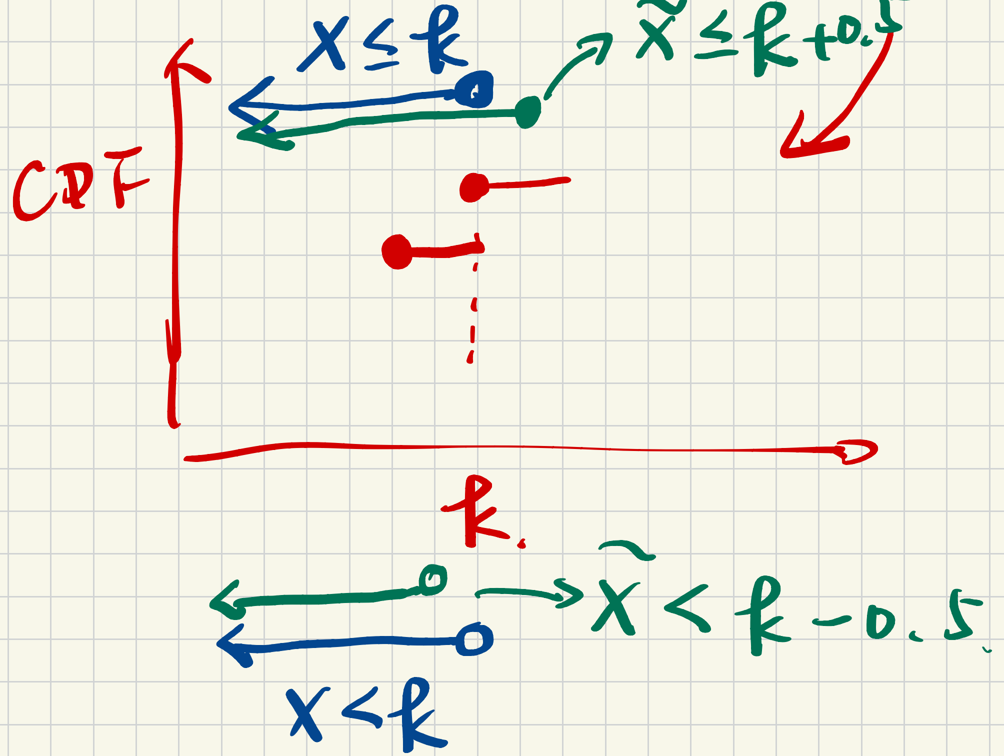
- $F_X(2) = F_X(2.1) = F_X(2.9)$
- $F_{\tilde{X}}(2) \neq F_{\tilde{X}}(2.9)$
- How should we approximate?

  $F_X(k) \approx F_{\tilde{X}}(k+0.5)$
- $P\{X \leq k\} \approx P\{\tilde{X} \leq k+0.5\}$
- $P\{X < k\} \approx P\{X \leq k-1\} \approx P\{\tilde{X} \leq k-0.5\}$
- $P\{X \geq k\} \approx 1 - P\{X < k\}$

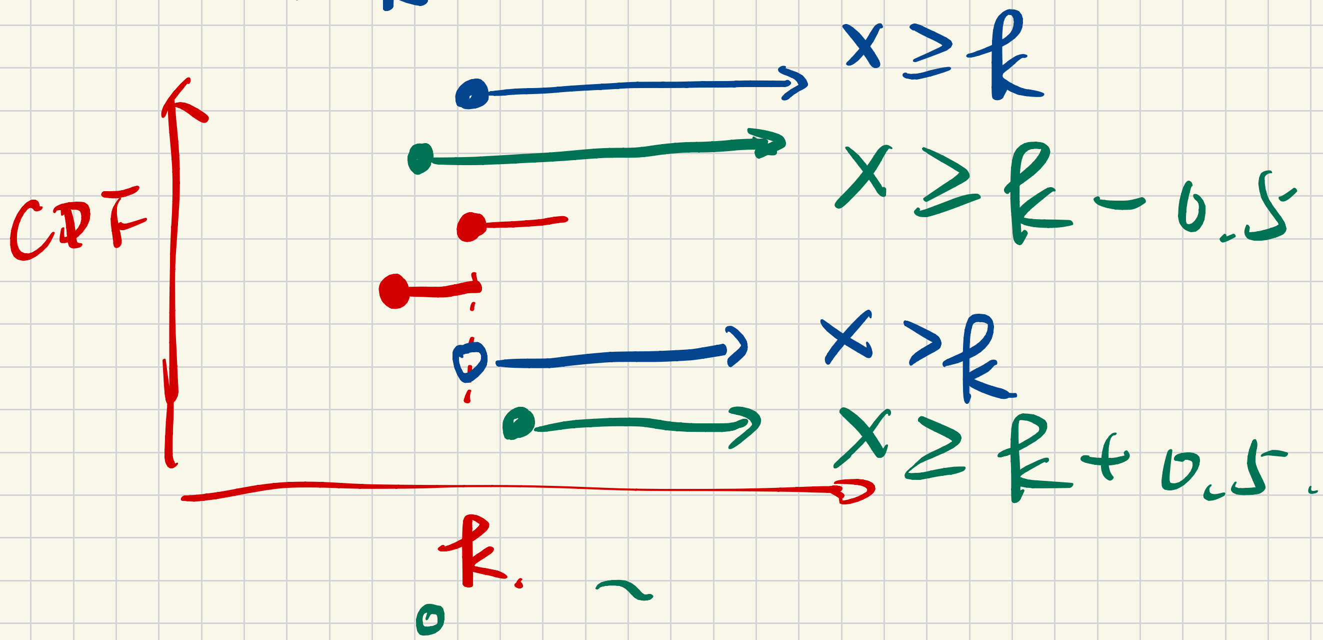
  $= P\{\tilde{X} > k-0.5\}$
- $P\{X > k\} \approx$

$P\{X \geq k+1\} = P\{\tilde{X} \geq k+0.5\}$





$X \le k$
$\tilde{X} \le k+0.5$
CDF
k
$\tilde{X} < k-0.5$
$X < k$



$X \ge k$
$X \ge k-0.5$
CDF
$X > k$
$X \ge k+0.5$
k

Standardized Binomials

Standardized $S_{n,p} \sim Bin(n,p)$

- $X = \frac{S_{n,p} - np}{\sqrt{np(1-p)}}$

DeMoivre-Laplace limit theorem (First CLT version)

$$\lim_{n\to\infty} \underbrace{P\left\{\frac{S_{n,p} - np}{\sqrt{np(1-p)}} \le c\right\}}_{\text{standardized binomial cdf.}} = \underbrace{\Phi(c)}_{\text{Normal dist. cdf.}}$$

Slido

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Each user independently opens today's push notification with probability $p = 0.3$.

- You send it to $n = 200$ users
- Let X be the number of opens
- $P\{X < 70\}$ in terms of Φ?

$(a)\ \Phi\left(\frac{70 - \mathrm{np}}{\sqrt{np(1-p)}}\right)$

$(b)\ \Phi\left(\frac{69.5 - \mathrm{np}}{\sqrt{np(1-p)}}\right)$

$(c)\ \Phi\left(\frac{70.5 - \mathrm{np}}{\sqrt{np(1-p)}}\right)$

$(d)\ \Phi\left(\frac{70}{\sqrt{np(1-p)}}\right)$