# Last lecture

Poisson process (Ch 3.5)

- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

# Agenda

**Gaussian (normal) Distribution** (Ch 3.6.2)

- Motivation and Definition
- Examples

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Definition
- CDF Approximation
- Examples

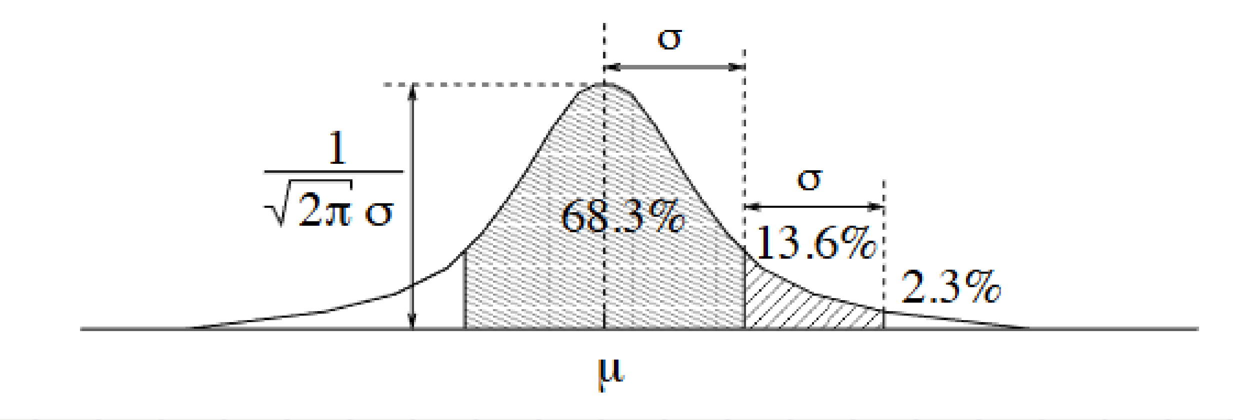
# Gaussian (Normal) Distribution

# Definition

A normal distribution is defined by $\mu_X$ and $\sigma_X^2$, Let $X \sim N(\mu, \sigma^2)$

- $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

- Usage – model "Sum of many small independent events"
- E.g. Sum of many binomial distributions



# Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$$

- $\Phi(u) \triangleq F_X(u) =$
- $Q(u) = 1 - \Phi(u)$
- Pre-computed tables!

# $\Phi$ and $Q$ tables

| *z* | −0.00 | −0.01 | −0.02 | −0.03 | −0.04 | −0.05 | −0.06 | −0.07 | −0.08 | −0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **-3.9** | 0.00005 | 0.00005 | 0.00004 | 0.00004 | 0.00004 | 0.00004 | 0.00004 | 0.00004 | 0.00003 | 0.00003 |
| **-3.8** | 0.00007 | 0.00007 | 0.00007 | 0.00006 | 0.00006 | 0.00006 | 0.00006 | 0.00005 | 0.00005 | 0.00005 |
| **-3.7** | 0.00011 | 0.00010 | 0.00010 | 0.00010 | 0.00009 | 0.00009 | 0.00008 | 0.00008 | 0.00008 | 0.00008 |
| ⋮ | | | | | | | | | | |
| **−0.1** | 0.46017 | 0.45620 | 0.45224 | 0.44828 | 0.44433 | 0.44038 | 0.43644 | 0.43251 | 0.42858 | 0.42465 |
| **−0.0** | 0.50000 | 0.49601 | 0.49202 | 0.48803 | 0.48405 | 0.48006 | 0.47608 | 0.47210 | 0.46812 | 0.46414 |
| *z* | **−0.00** | **−0.01** | **−0.02** | **−0.03** | **−0.04** | **−0.05** | **−0.06** | **−0.07** | **−0.08** | **−0.09** |

| *z* | + 0.00 | + 0.01 | + 0.02 | + 0.03 | + 0.04 | + 0.05 | + 0.06 | + 0.07 | + 0.08 | + 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.50000 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.52790 | 0.53188 | 0.53586 |
| **0.1** | 0.53983 | 0.54380 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56360 | 0.56749 | 0.57142 | 0.57535 |
| ⋮ | | | | | | | | | | |
| **3.8** | 0.99993 | 0.99993 | 0.99993 | 0.99994 | 0.99994 | 0.99994 | 0.99994 | 0.99995 | 0.99995 | 0.99995 |
| **3.9** | 0.99995 | 0.99995 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99997 | 0.99997 |
| *z* | **+0.00** | **+0.01** | **+0.02** | **+0.03** | **+0.04** | **+0.05** | **+0.06** | **+0.07** | **+0.08** | **+0.09** |

# Scaling the Gaussian RV

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$$

- $Y = \sigma X + \mu$
- $f_Y(y) =$

# Examples

Given $\Phi(u) \triangleq F_X(u)$ and $Q(u) = 1 - \Phi(u)$ for $X \sim N(0,1)$

- Let $Y = N(\mu = 10,\ \sigma^2 = 16)$
- Find $P\{Y > 15\}, P\{Y \leq 5\}, P\{Y^2 \geq 400\}$ and $P\{Y = 2\}$ in terms of $\Phi$ or $Q$

# Examples

Suppose $\mu_X = 10$ and $\sigma_X^2 = 3$. Compute $P\{X < 10 - \sqrt{3}\}$ if

- $X$ is a Gaussian RV in terms of $Q$
- $X$ is a uniform RV

(Hint: $10 - \sqrt{3} \approx 8.27$)

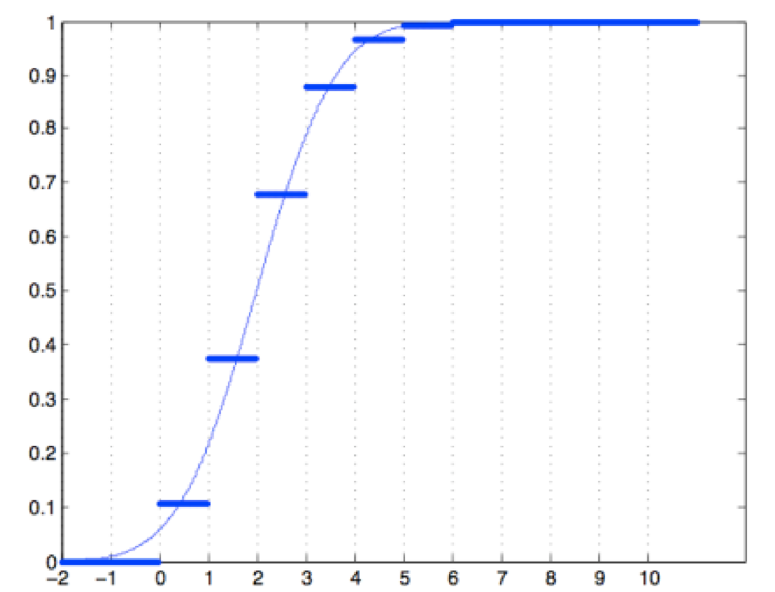
# Central Limit Theorem and Gaussian Approximation

# Central Limit Theorem (CLT)

If many independent random variables are added together, and if each of them is small in magnitude compared to the sum, then the sum $X$ has an approximately Gaussian distribution $\tilde{X}$.

- $P\{X \leq v\} \approx P\{\tilde{X} \leq v\}$
- E.g., $X \sim Binomial(n, p)$ when $np$ and $n(1-p)$ are not small
  - $(n, p) = (10, 0.2)$
  - $\mu_X =$
  - $\sigma_X^2 =$

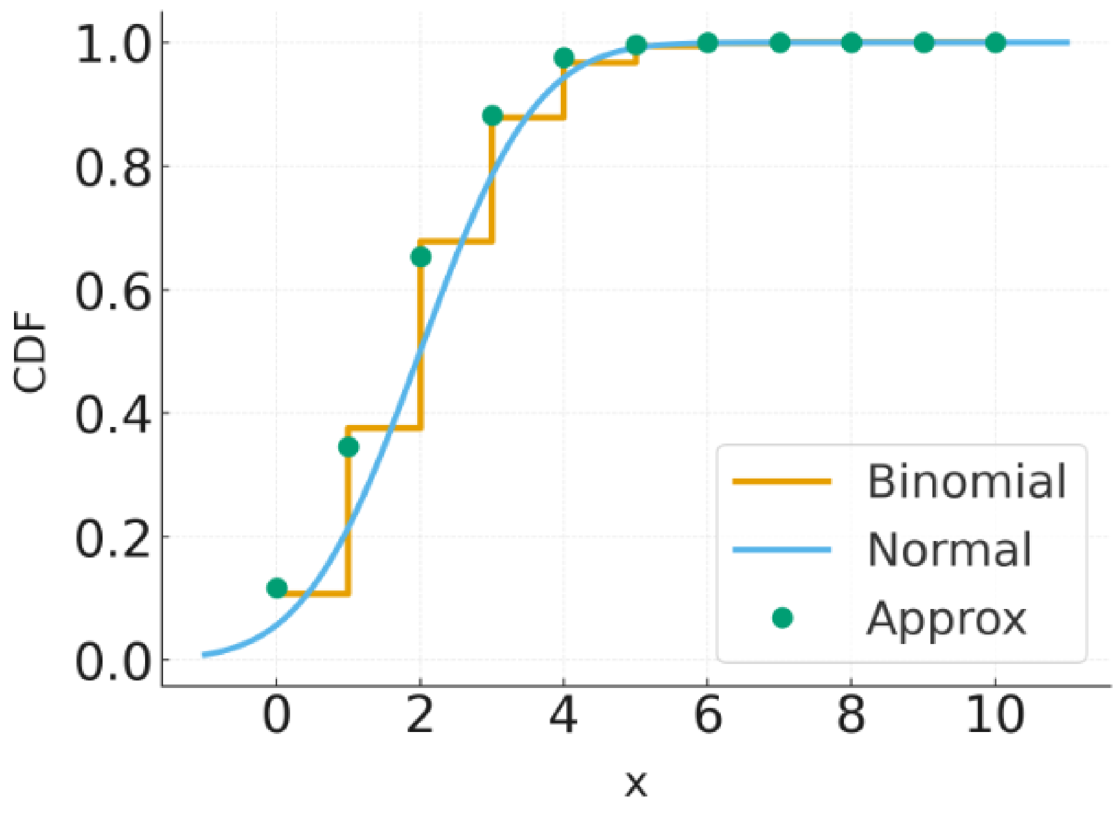
  - What if $np$ is small?



# Gaussian Approximation - 1

Approximate $X \sim Bin(10, 0.2)$ with $\tilde{X} \sim N(2, 1.6)$

- $F_X(2) = F_X(2.1) = F_X(2.9)$
- $F_{\tilde{X}}(2) \neq F_{\tilde{X}}(2.9)$
- How should we approximate?




- $P\{X \leq k\} \approx$
- $P\{X < k\} \approx$
- $P\{X \geq k\} \approx$
- $P\{X > k\} \approx$