Last lecture

Poisson process (Ch 3.5)

- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

Agenda

Gaussian (normal) Distribution (Ch 3.6.2)

- Motivation and Definition
- Examples

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Definition
- CDF Approximation
- Examples

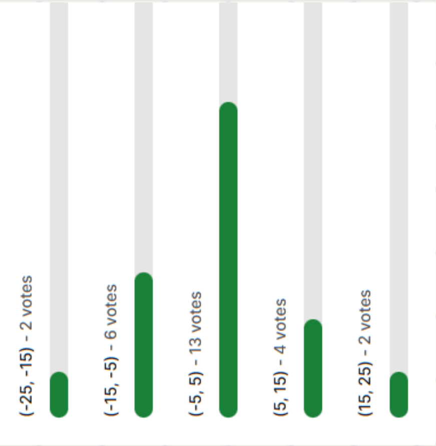
Gaussian (Normal) Distribution

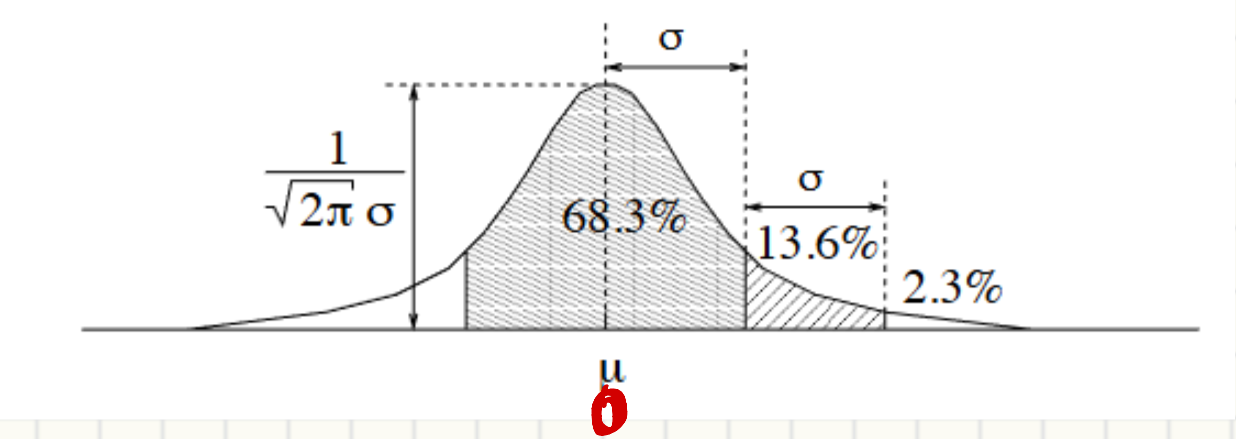
Definition

A normal distribution is defined by μ_X and σ_X^2, Let $X \sim N(\mu, \sigma^2)$

- $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

- Usage – model “Sum of many small independent events”
- E.g. Sum of many binomial distributions



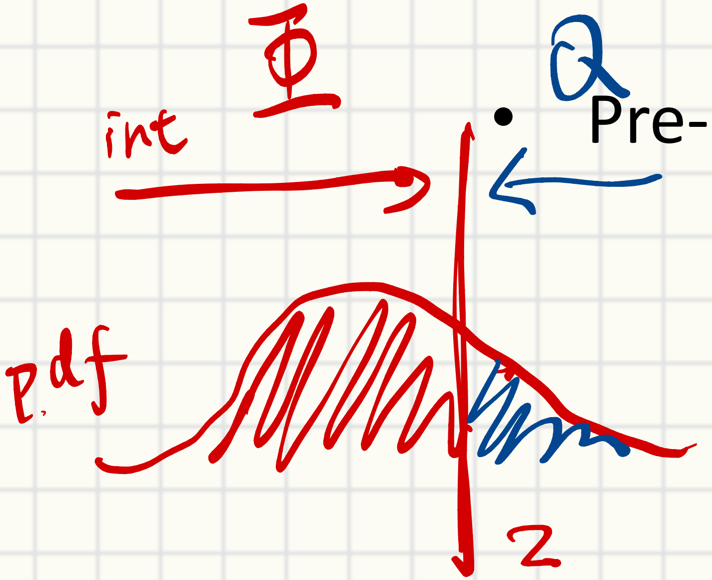


Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$

- $\Phi(u) \triangleq F_X(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$
- $Q(u) = 1 - \Phi(u)$ $= F_X^c(u)$
- Pre-computed tables!



Eg. $P\{X \le 0.5\}$

$= \Phi(0.5)$

$P\{X \ge 2\} = Q(2)$

Φ and Q tables

$P\{X \le -3.77\} = 8 \times 10^{-5}$

z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008

$P\{X \le 0\} = 0.5$

⋮

−0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
−0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535

⋮

$P\{X \le 0.106\}$

3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09

Scaling the Gaussian RV

$$\boxed{f(u) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{u^2}{2})}$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$$

- $Y = \sigma X + \mu$ $\sim N(\mu, \sigma^2)$
- $f_Y(y) =$

$$f_X\left(\frac{y-\mu}{\sigma}\right)\cdot\frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

$$E[Y] = \sigma \underbrace{E[X]}_{0} + \mu$$

$$= \mu$$

$$\sigma_Y^2 = \sigma^2\sigma_X^2$$

$$= \sigma^2$$

Connecting Gaussian to Exponential

Exam Safe

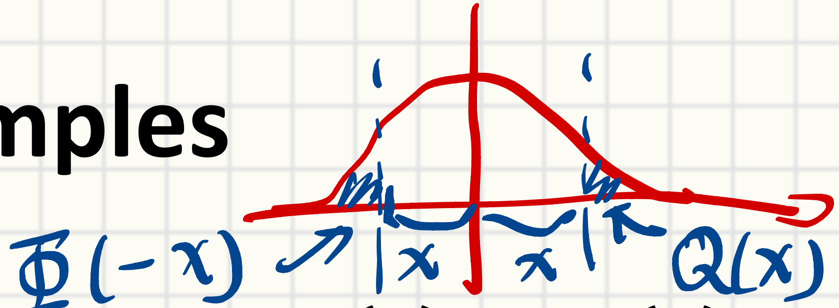
Comparing Normal $Z \sim N(0,1)$ with Exponential $E(\lambda = \frac{1}{2})$

- $f_Z(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})$ $f_E(u) = \frac{1}{2} \exp(-\frac{u}{2})$ $f_E = \lambda e^{-\lambda t}$
- Let $X = Z^2$ (Chi-square)
- $f_X(u) \propto f_Z(\sqrt{u}) + f_Z(-\sqrt{u}) \propto \exp\left(-\frac{u}{2}\right) \sim E(\lambda = \frac{1}{2})$

Example – Circuit voltage noise follows $Z \sim N$

- Energy is Z^2 - cause system fail

Examples



$\Phi(x) = Q(-x)$

Given $\Phi(u) \triangleq F_X(u)$ and $Q(u) = 1 - \Phi(u)$ for $X \sim N(0,1)$

- Let $Y = N(\mu = 10,\ \sigma^2 = 16)$ $= Y = 4X+10$
- Find $P\{Y > 15\}, P\{Y \le 5\}, P\{Y^2 \ge 400\}$ and $P\{Y = 2\}$ in terms of Φ or Q

$P\{Y > 15\} = P\{4X+10 > 15\} = P\{X > \frac{15-10}{4}\}$

$= P\{X > 1.25\}$

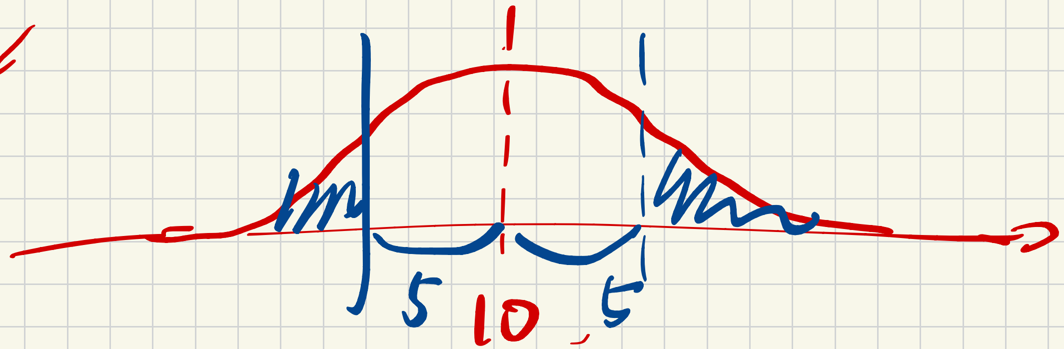
$= Q(1.25)$

$P\{Y \le 5\} = P\{X \le \frac{5-10}{4}\}$

$= P\{X \le -1.25\} = \Phi(-1.25)$

$= Q(1.25)$

PY



$$P\{Y^2 \geq 400\} = P\{Y \leq -20\} + P\{Y \geq 20\}$$

$$= P\left\{X \leq \frac{-20-10}{4}\right\} + P\left\{X \geq \frac{20-10}{4}\right\}$$

$$= \underbrace{\Phi(-7.5)}_{<10^{-6}} + Q(2.5) \approx Q(2.5)$$

Examples

Suppose $\mu_X = 10$ and $\sigma_X^2 = 3$. Compute $P\{X < 10 - \sqrt{3}\}$ if

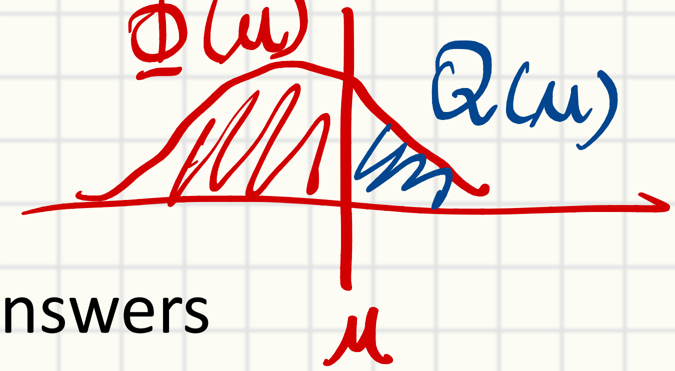
- X is a Gaussian RV in terms of Q
- X is a uniform RV

(Hint: $10 - \sqrt{3} \approx 8.27$)

Slido

$\Phi = F_X$

$Q = F_X^c$



Choose all the correct answers

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(a) $\Phi(u) + Q(u) \cancel{>} 1$ ✓

$=$

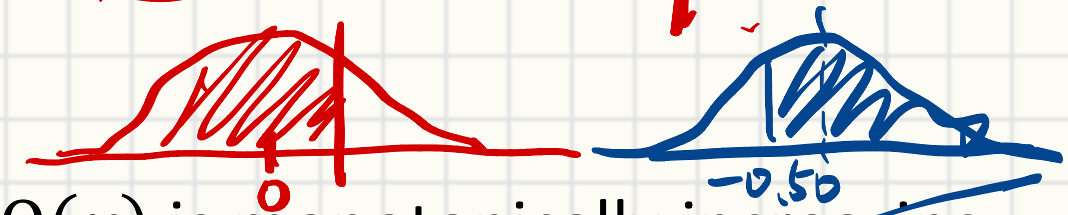
(b) $\Phi(u) - Q(-u) = 0$ ✓

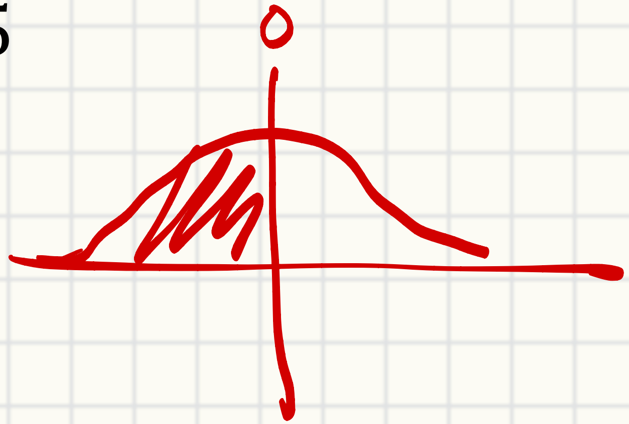
$\Phi(u) = Q(-u)$

>0.5 $\quad$ >0.5

(c) $\Phi(0.5) + Q(-0.5) > 0$ ✓ $\quad 1$

(d) $\Phi(0) = 0.5$ ✓





(e) $Q(x)$ is monotonically ~~increasing~~

decreasing