Last lecture

Poisson process (Ch 3.5)

- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

Agenda

Poisson process (Ch 3.5) Examples

Erlang Distribution (Ch 3.5.3) Definition

Linear Scaling (Ch 3.6.1)

- Equation and derivation
- Examples

Gaussian (normal) Distribution (Ch 3.6.2)

- Motivation and Definition

Example

Calls arrive to a support center at rate $\lambda = 2$ calls per minute.
Let N_t denotes the number of calls until time t (mins)

- $N_t \sim$ Poi $(\lambda = 2t)$
- $P_{N_t}(k) =$ $\dfrac{e^{-2t}(2t)^k}{k!}$
- $P\{$ No calls arrive in the first 3.5 minutes $\}$
- $P\{$ The third call arrives after time t = 3.$\}$

$$P_{N_{3.5}}(0) = \frac{e^{-7}\,7^0}{0!} = e^{-7}$$

$$\sum_{k=0}^{2} P_{N_3}(k)$$

Erlang Distribution

Definition

Let T_r denotes the time of r^{th} count of a Poisson process

- $T_r = \sum_{i=1}^{r} U_i \, , U_i \sim Exp(\lambda)$
- $F_{T_r}^{c}(\mathrm{t}) = \mathrm{P}\{\mathrm{T_r} > \mathrm{t}\}$: "At most $r - 1$ count by time t"
- $F_{T_r}^{c}(\mathrm{t}) = \sum_{k=0}^{r-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ $\quad X \sim Poi(\lambda t)$
- $f_{T_r}(t) = -\frac{dF_{T_r}^{c}(t)}{dt} = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!} = \underbrace{P_X(r-1)}_{r-1 \text{ fails by } t} \cdot \underbrace{\lambda}_{\text{fail at } t.}$
- Intuitively – Exactly $r-1$ arrival before time t, and the r^{th} event comes at t

Linear Scaling

Definition

$a=3 \quad b+2 \qquad 3X+2 \quad X\in[0,1]$

$\downarrow$

$Y\in[3\times0+2,\ 3\times1+2] = [2,5]$

Let $Y = aX + b$, where X, Y are RV and a, b are constants

- $\boxed{f_Y(u) = f_X\left(\frac{u-b}{a}\right) \times \frac{1}{a}}$
- Example - $X \sim Uniform(0,1)$

$f_X(v)=1 \quad 0\le v\le 1$

$f_Y(u) = \frac{1}{a} \quad b\le u\le \cancel{b+1}\ a+b$

- $F_Y(u) = P\{aX + b < u\} = P\{X < \frac{u-b}{a}\} = F_X\left(\frac{u-b}{a}\right)$
- $f_Y(u) = F_Y'(u) = F_X'\left(\frac{u-b}{a}\right) = f_X\left(\frac{u-b}{a}\right)\cdot\frac{1}{a}$
- $E[Y] =$ $aE[X]+b$
- $Var(Y) =$ $a^2\,Var(X)$ $\sigma_Y =$ $a\,\sigma_X$

Example

Consider the temperature in Champaign

- X denotes the temperature in C (Celcius)
- Y denotes it in F (Fahrenheit)
- $Y = (1.8)X + 32$

  $a = 1.8 \quad b = 32$

  $f_Y(y) = f_X\left(\frac{y-32}{1.8}\right) \cdot \frac{1}{1.8}$
- Express f_Y in terms of f_X
- Find f_Y if $X \sim Uniform(15, 20)$

  $f_X = \frac{1}{5} \quad 15 \leq x \leq 20$

$f_Y = \frac{1}{9} \qquad 59 \leq y \leq 68$

27   36

32   32

Example

$Z = \quad \nearrow \mu_z = 0$

Let $X \sim Uniform(a,b)$. Find the pdf of standardized RV $\frac{X-\mu_X}{\sigma_X}$

$\searrow \sigma_z = 1$

- Recall the variance of $Uniform(0,1)$ is $\cancel{12}$ $\frac{1}{12}$

$Z = \frac{1}{\text{seg.}}$

$\underbrace{z = [-k, k]}_{} \rightarrow 2\sqrt{3}$

$k = \sqrt{3}$

$k - (-k) = 2k$

$1 \cdot (2k)^2 = 12$

$f_z = \frac{1}{2\sqrt{3}} \qquad z = [-\sqrt{3}, \sqrt{3}]$

$k = \sqrt{3}$

Gaussian (Normal) Distribution

Definition

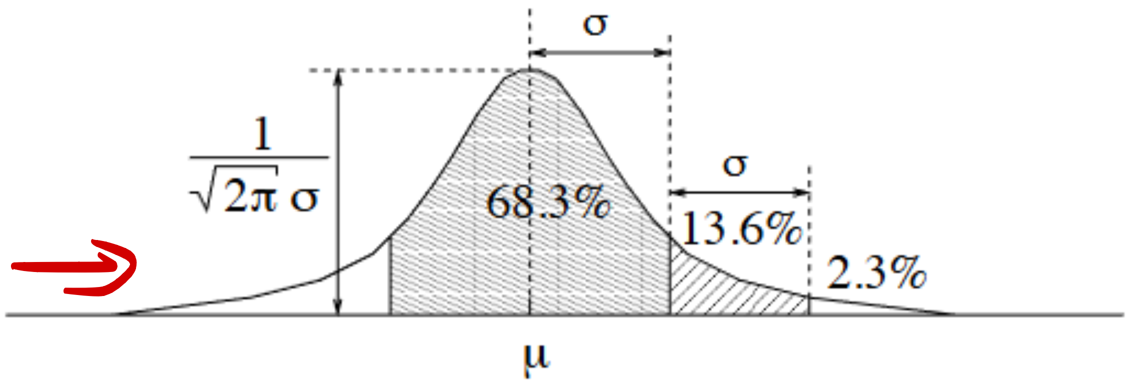
A normal distribution is defined by μ_X and σ_X^2, Let $X \sim N(\mu, \sigma^2)$

- $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

follow any distributions

- Usage – model “Sum of many small independent events”
- E.g. Sum of many binomial distributions

Bell shape →



Each of you pick $[1, 10] = A$

Find 3 neighbors, for each $(\overset{A}{\text{you}}, \overset{B}{\text{neighbor}})$ pair.

$$\text{Score} = \|A - B\| - \|(A+B) - 12\|$$

min score. $A = B = 1$ $\quad S = -10$

max score $A = 2$, $B = 10$ $\quad S = 8$

sum scores

$$S \in [-30, 24]$$