Last lecture

Uniform Distribution (Ch 3.3)

- Definition
- Properties

Exponential Distribution (Ch 3.4)

- Definition
- Properties
- Connection with Geometric RV

Agenda

Poisson process (Ch 3.5)

- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

Erlang Distribution (Ch 3.5.3)

- Definition
- Properties

Waiting for the bus

Say a bus comes to the stop every 10 minutes in average

- Expected waiting time follows $T \sim Exp(\lambda = \frac{1}{10})$
- Alice knows the time last bus leave at $t = 0$, what's her best strategy to arrive t^* at the stop minimizing her waiting time T'?

$$T' \begin{cases} s - t^* & \text{if } s > t^* \quad f_T(s) \\ E[T] = \frac{1}{\lambda} & \text{else } (s < t^*) \quad F(t^*) \end{cases}$$

$$E[T'] = \int_{t^*}^{\infty} (s - t^*) f_T(s)\, ds + \frac{1}{\lambda} \cdot F(t^*) = \frac{1}{\lambda} = 10 \text{mins}$$

Poisson Process

Motivation

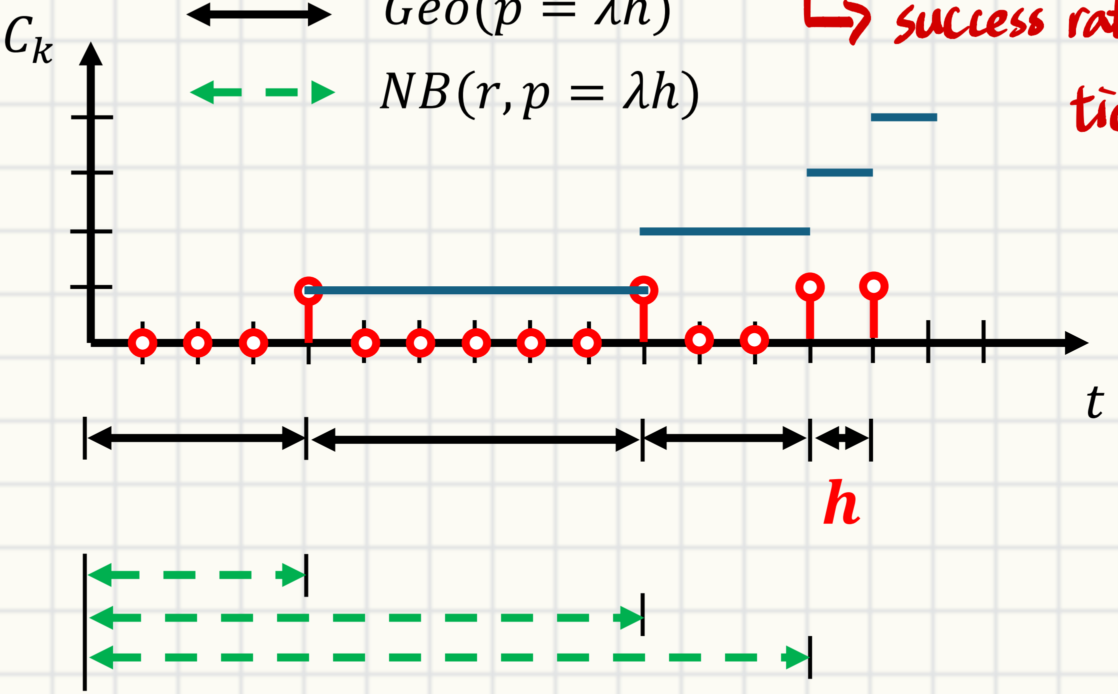
- Model the process of customer coming inside a coffee shop
- The process of incoming calls for customer service
  - If a customer (call) only comes at the start of every minutes -> Bernoulli process ⇒ Every mins we do a trial.
  - But what if we want to model the time
  - Define a small h between Bernoulli trials

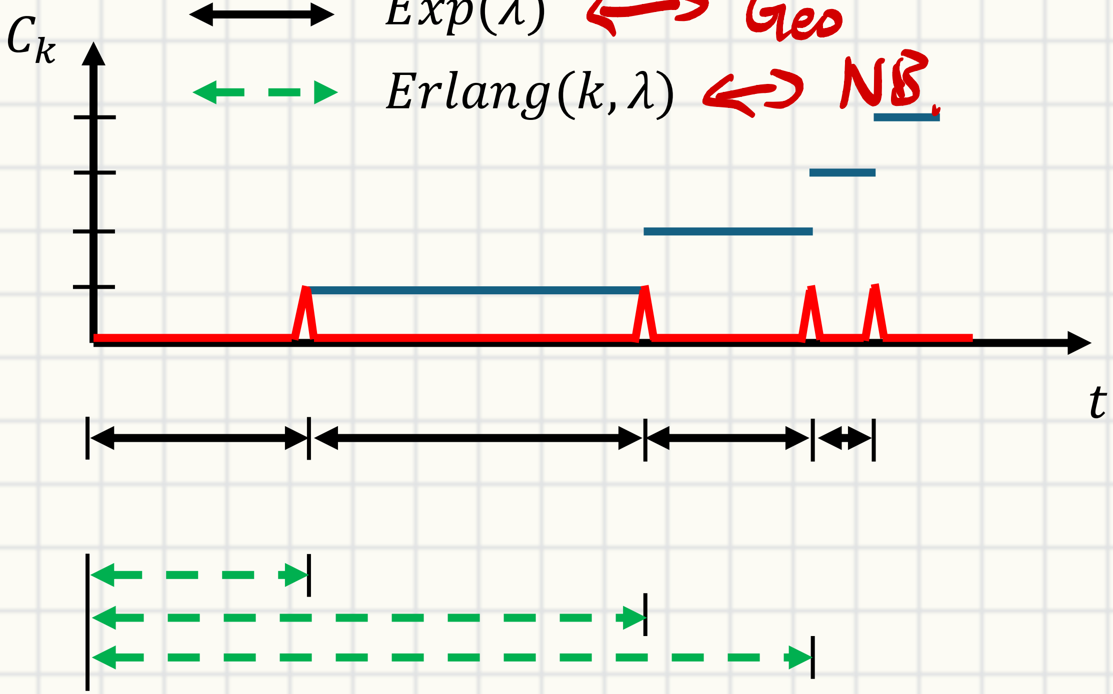
Bernoulli Process $\xrightarrow{h \to 0,\, \lambda = \frac{p}{h}}$ Poisson Process

- Assume each trial takes h time to complete

$\boxed{\text{discrete}}$

Bernoulli		$h \to 0, \lambda = \frac{p}{h}$	Poisson
∘	$Bern(p)$		
—	$Bin\left(n = \left\lfloor \frac{t}{h} \right\rfloor, p = \lambda h\right)$ ↳ success rate per tick.	$\longrightarrow$	— $Poi(\lambda = \lambda t)$ (param) $\lambda = np$
⟷	$Geo(p = \lambda h)$		⟷ $Exp(\lambda) \longleftrightarrow$ Geo
⇠⇢	$NB(r, p = \lambda h)$		⇠⇢ $Erlang(k, \lambda) \longleftrightarrow$ NB





Definition

A Poisson process with rate λ is a random counting process $N = (N_t : t \geq 0)$ s.t.

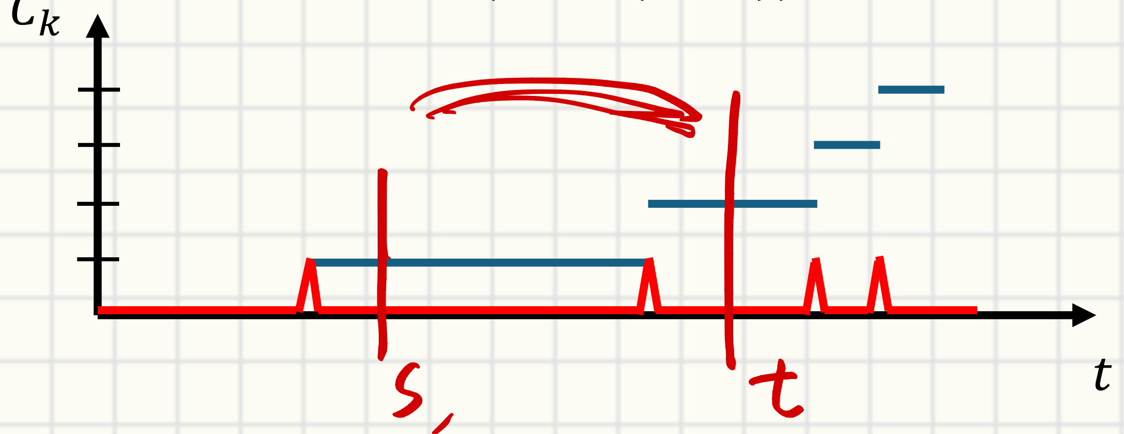
- $N_t - N_s$ follows Poisson distribution $Poi(\lambda = \lambda(t-s))$
- For $0 \leq t_1 \leq t_2 \ldots \leq t_k$, $N_{t_k} - N_{t_{k-1}}$ the increments are independent with each other

Coffee analogy

λ

$s = 10$ AM $T = 11$ AM $\lambda = 2$ customers per 10 mins

$Poi(\lambda = \lambda(t-s))$



customers $[s, T]$

$X \sim Poi(\lambda = 2 \cdot 6)$

Example

$$P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Consider a Poisson $\boxed{\text{process}}$ with rate $\lambda > 0$ in time interval $[0, T]$

- X is the total number of count during $[0, T]$
- X_1 is the count during $[0, \tau]$, $0 < \tau < T$
- X_2 is the count during $[\tau, T]$

$X_2 \sim Poi(\lambda = \lambda(T - \tau))$

- Solve $P\{X = n\}$, $P\{X_1 = i\}$ and $P\{X_2 = j\}$

$X \sim Poi(\lambda T)$

$$P_X(n) = \frac{(\lambda T)^n e^{-(\lambda T)}}{n!}$$

$$\frac{(\lambda \tau)^i e^{-(\lambda \tau)}}{i!}$$

$$\frac{[\lambda(T-\tau)]^j e^{-[\lambda(T-\tau)]}}{j!}$$

Example

Consider a Poisson process with rate $\lambda > 0$ in time interval $[0, T]$

- X is the total number of count during $[0, T]$
- X_1 is the count during $[0, \tau]$, $0 < \tau < T$
- X_2 is the count during $[\tau, T]$

- Let $n = i + j$
- Solve $P(X = n | X_1 = i)$

$$= P((X_1 = i, X_2 = j) | X_1 = i)$$

independent.

$$= P(X_2 = j) = \frac{[\lambda(T-\tau)]^j e^{-[\lambda(T-\tau)]}}{j!}$$

Example

Consider a Poisson process with rate $\lambda > 0$ in time interval $[0, T]$

- X is the total number of count during $[0, T]$
- X_1 is the count during $[0, \tau]$, $0 < \tau < T$
- X_2 is the count during $[\tau, T]$

$P(X_1 = i, X_2 = j)$

- Let $n = i + j$
- Solve $P(X_1 = i | X = n)$ $= \dfrac{P(X_1 = i, X = n)}{P(X = n)}$

$$= \frac{n!}{i!\,j!} \left(\frac{\tau}{T}\right)^i \left(\frac{T-\tau}{T}\right)^j = \binom{n}{i} p^i (1-p)^j$$

Example

Calls arrive to a support center at rate $\lambda = 2$ calls per minute.
Let N_t denotes the number of calls until time t (mins)

- $N_t \sim$ $\text{Poi}(\lambda = 2t$
- $P_{N_t}(k) =$ $\frac{(2t)^k e^{-2t}}{k!}$
- $P\{$ No calls arrive in the first 3.5 minutes $\}$ $k=0$
- $P\{$ The third call arrives after time t = 3.$\}$

$$\frac{(7)^0 e^{-7}}{0!} = e^{-7}.$$