Last lecture

Cumulative Distribution Function (Ch 3.1)

- Examples
- CDF to PMF and probabilistic density function (PDF)

Continuous RV & Probability Density Function (Ch 3.2)

- Definition
- Facts

Agenda

Uniform Distribution (Ch 3.3)

- Definition
- Properties

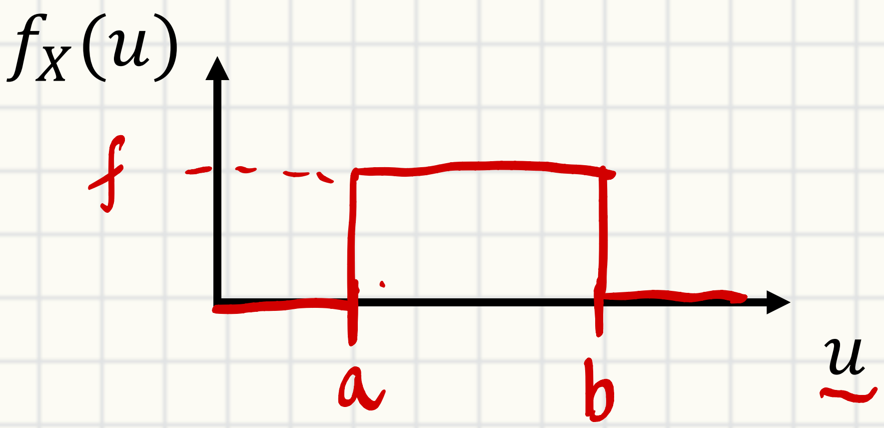
Exponential Distribution (Ch 3.4)

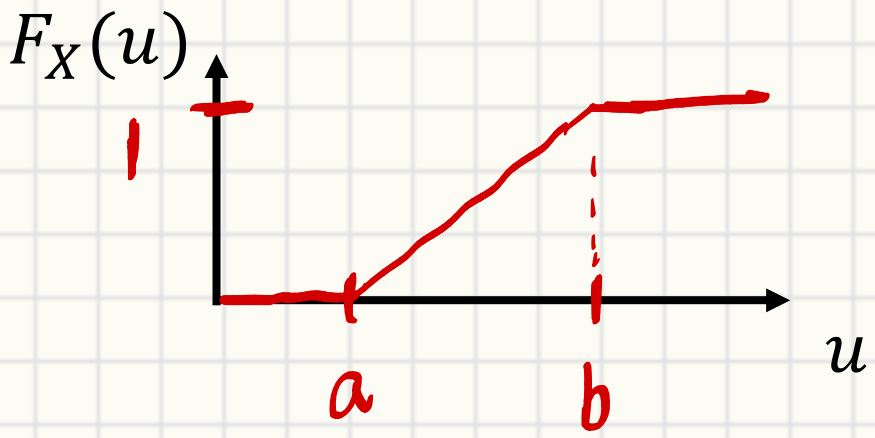
- Definition
- Properties
- Connection with Geometric RV

Uniform Distribution

Uniform Distribution

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \le u \le b \\ 0 & \text{else} \end{cases}$$





Area Under Curve (AUC) = 1

$\int_a^b f\,du = 1 \qquad f \cdot u\Big|_a^b = f(b-a) = 1$

Properties

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \le u \le b \\ 0 & \text{else} \end{cases}$$

- $E[X] = \int_{-\infty}^{\infty} u f_X(u)\, du = \int_a^b \frac{u}{b-a}\, du = \frac{u^2}{2}\Big|_a^b \left(\frac{1}{b-a}\right) = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{b+a}{2}}$ μ_X
- $E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\, du = \int_a^b \frac{u^2}{b-a}\, du = \boxed{\frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}}$ $E[X^2]$
- $Var(X) = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \boxed{\frac{(a-b)^2}{12}}$ σ_X^2
- Special case, when $(a, b) = (0,1)$
  - k^{th} moment $E[X^k] = \frac{1}{k+1}$
  - $Var(X) = \frac{1}{12}$

Exponential Distribution

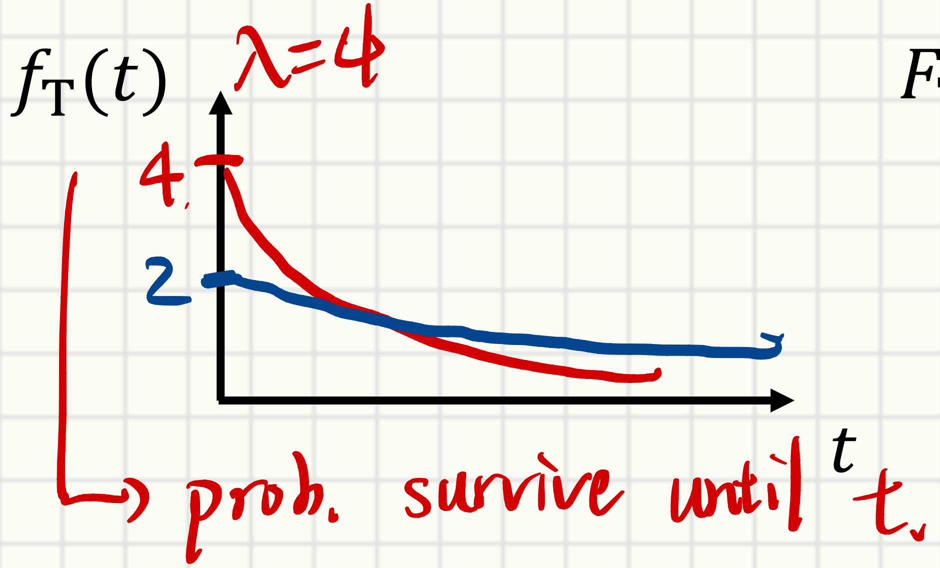
Exponential Distribution

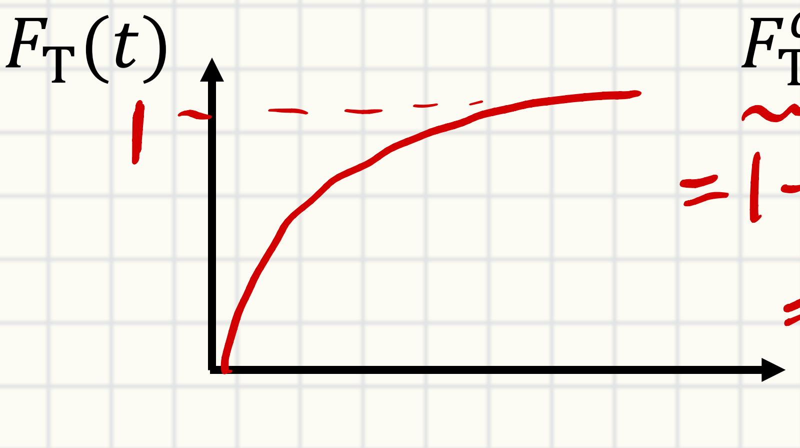
Motivation – System life for failure rate λ $\quad T \sim \text{Exp}(\lambda)$

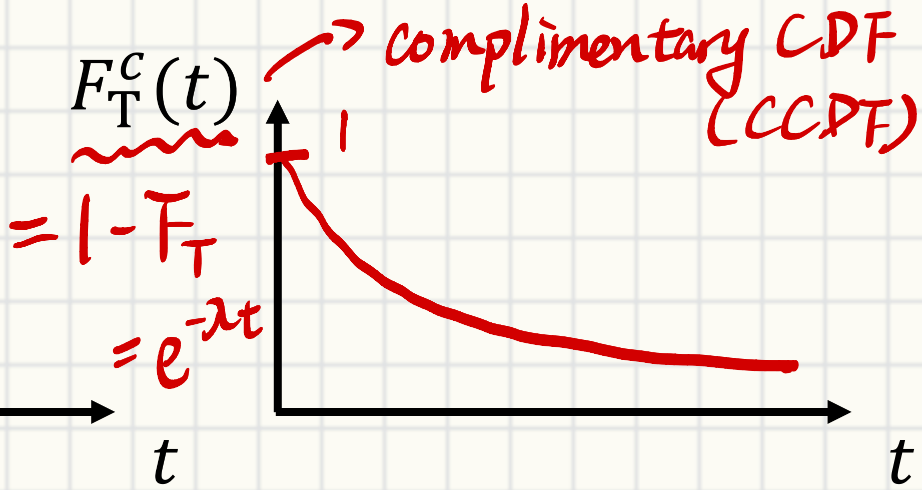
$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$= \int_0^t f_T(u)\,du = -e^{-\lambda t} + C.$$







Properties

$\int u\,dv = uv - \int v\,du$

$u = t^n \quad v = -e^{-\lambda t}$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

- $$E[T^n] = \int_0^\infty t^n \overbrace{\lambda e^{-\lambda t}}^{f_T(t)} dt$$
  $$= -t^n e^{-\lambda t} \Big|_0^\infty + \int_0^\infty n t^{n-1} e^{-\lambda t} dt$$ Int. by part.
  $$= 0 + \frac{n}{\lambda} \int_0^\infty t^{n-1} \lambda e^{-\lambda t} dt = \frac{n}{\lambda} E[T^{n-1}]$$
- $E[T] = \frac{1}{\lambda}$ $\qquad E[T^2] = \frac{2}{\lambda^2}$ $\qquad E[T^n] = \frac{\cancel{n}\ n!}{\lambda^n}$
- $Var(T) = E[T^2] - \mu_T^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Examples

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Let $T \sim Exp(\lambda = ln2)$, find $P\{T \geq t\}$ and $P(T \leq 1 | T \leq 2)$ $\to P(A|B)$

$= \frac{P(AB)}{P(B)}$

$F_T^c(t) = 1 - F_T(t)$

$= 1 - P\{T \leq t\}$

$P\{T \geq t\} = F_T^c(t) = e^{-\lambda t}$

$= e^{-ln2 t} = 2^{-t}$

$P\{T \leq 1, T \leq 2\}$

$= \frac{1 - e^{-\lambda 1}}{1 - e^{-\lambda 2}} = \frac{1 - \frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$

Memoryless Property

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$P\{T \geq t\} = e^{-\lambda t}$$

- $P\{T \geq s + t | T \geq s\} =$ $P\{T \geq t\}$

- If T is the system lifetime

$$P\{T \geq s+t\} = e^{-\lambda(s+t)}$$

$$P\{T \geq s\} = e^{-\lambda s}$$

$$P\{T \geq s+t \mid T \geq s\} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

Connecting Exp with Geo

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

hours

sec $= \frac{1}{3600}$ h.

Summary - $F_L(\lfloor \frac{c}{h} \rfloor) \to F_T(c)$ when $h \to 0$

- $L \sim Geo(p = \lambda h)$
- $T \sim Exp(\lambda = \lambda)$

A lightbulb of average lifetime 1000hrs

- Failed hour = $L \sim Geo(p = \frac{1}{1000})$ why? $E[L] = \frac{1}{p} = 1000$
- Let's assume it will only fail at start of each ticks h hours (e.g., sec, $h = 1/3600$)
- Failed ticks = $L_h \sim Geo(p_h = \frac{1}{1000} \times h)$

Connecting Exp with Geo

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

A lightbulb of average lifetime 1000hrs

- Failed hour = $L \sim Geo(p = \frac{1}{1000})$
- Let's assume it will only fail at start of each ticks h hours (e.g., sec, $h = 1/3600$)
- Failed ticks = $L_h \sim Geo(p = \frac{1}{1000} \times h)$

hours we survive

seconds

- $P\{L_h h > c\} = P\left\{L_h > \frac{c}{h}\right\} = F^c_{L_h}\left(\left\lfloor \frac{c}{h} \right\rfloor\right) = (1-p)^{\left\lfloor \frac{c}{h} \right\rfloor}$

$$= (1 - \lambda h)^{\left\lfloor \frac{c}{h} \right\rfloor} = \lim_{h \to 0} -e^{-\lambda c} = F^c_T(c)$$

Slido – Waiting for the bus

#1438252

Say a bus comes to the stop every 10 minutes in average

- Expected ~~waiting~~ inter bus time follows $T \sim Exp(\lambda = \frac{1}{10})$
- Alice knows the time last bus leave at $t = 0$, what's her best strategy to arrive t^* at the stop minimizing her waiting time T'?

$E[T'] =$ catch

$(T - t^*) P\{t^* < U\}$

miss

(A) $t^* = 0$

(B) $t^* = 5$

(C) $t^* = 10$

(D) Doesn't matter