Last lecture

Reliability & Union Bound(Ch 2.12)

- Examples – network outage
- Examples – Array code

Exam Safe

Continuous RV (Ch 3)

- Motivation
- Cumulative Distribution Function (Ch 3.1)
- Examples

Agenda

Cumulative Distribution Function (Ch 3.1)

- Examples
- CDF to PMF and probabilistic density function (PDF)

Continuous RV & Probability Density Function (Ch 3.2)

- Definition
- Facts

Examples

$$P\{X=u\} = F_X(u) - F_X(u-)$$

- Find all u where $P\{X = u\} > 0$
- Find $P\{X \leq 0\}$
- Find $P\{X < 0\}$

jump at u

$u=0 \quad P\{X=0\}=0.5$

$$1 = F_X(0)$$

$$F_X(0) - P\{X=0\}$$

$$= 1 - 0.5 = 0.5$$

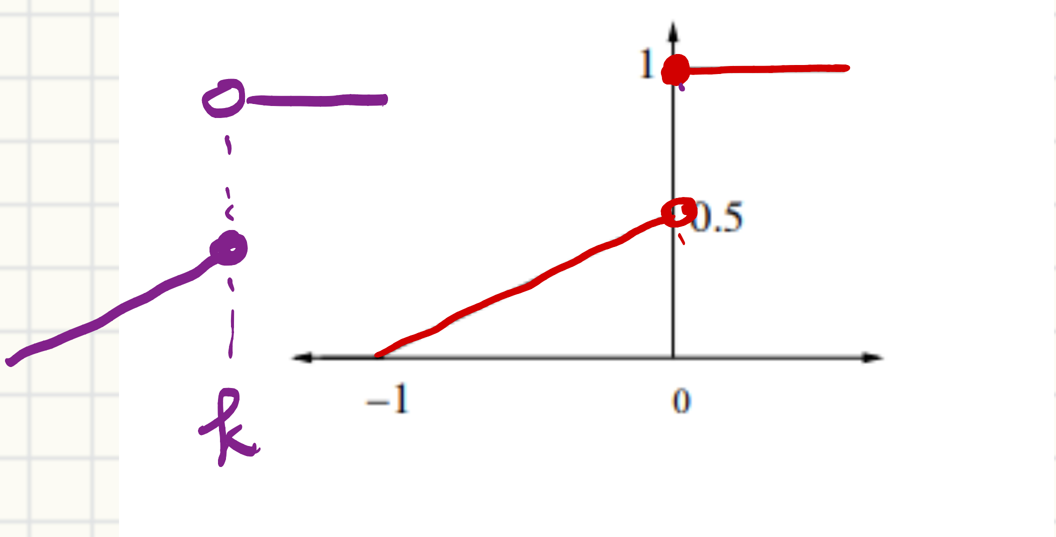


Figure 3.2: An example of a CDF.

CDF Properties

$P\{X \le u\}$

$F(u) = k$

$F(u+\Delta) = k + 0$

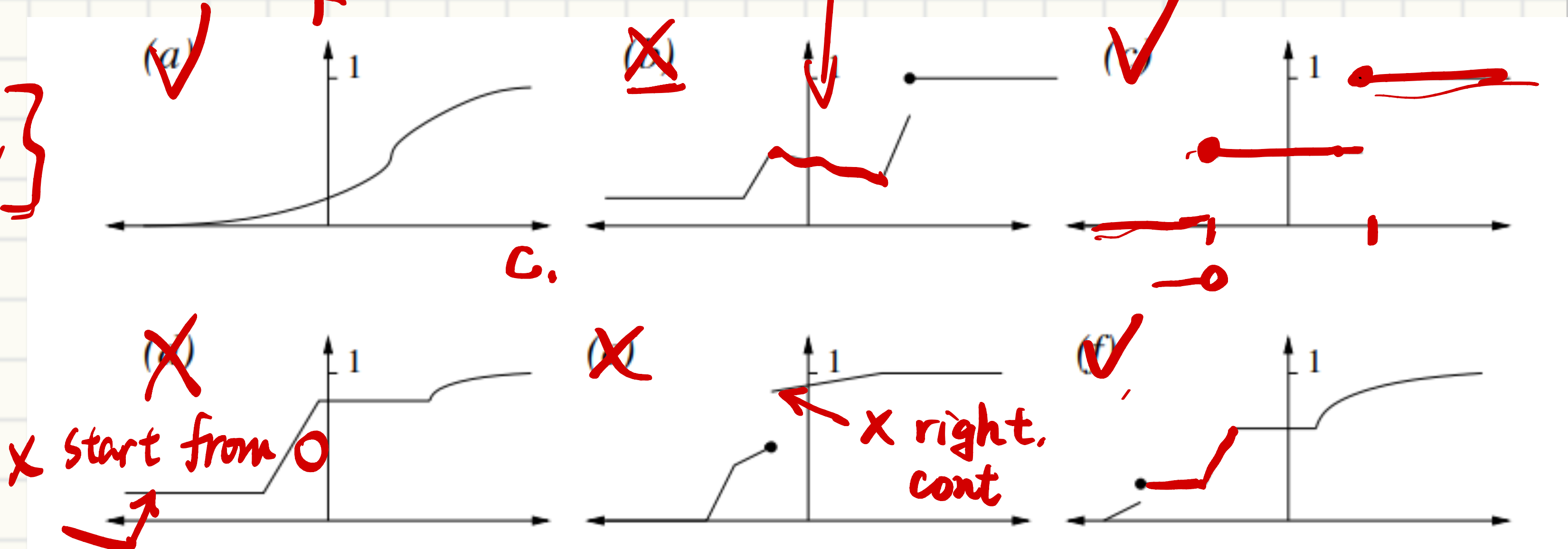
$P\{X \le u+\Delta\}$ at least k.

A function F is a CDF of some RV iif

- F is none-decreasing
- $\lim_{c\to\infty} F(c) = 1$ and $\lim_{c\to-\infty} F(c) = 0$
- F is right-continuous

decreasing.

$F(c) \triangleq P\{X \le c\}$



CDF to PMF and PDF

For discrete RV

- $F_X(c) = \sum_{u \le c} p_X(u)$
- $p_X(u) = F_X(u) - F_X(u-)$

before next possible outcome

For continuous RV

- $F_X(c) = \int_{-\infty}^{c} f_X(u)\,du$
- $f_X(u) = F_X'(u)$
- $f_X(u)$ is called "probability density function" (PDF)

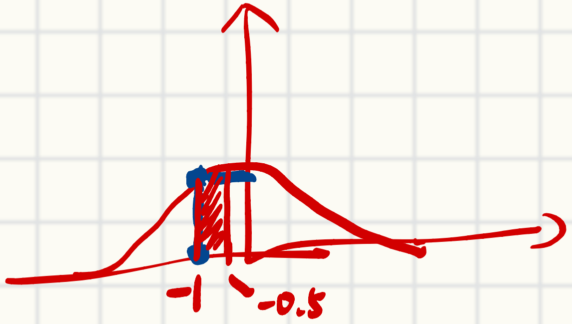
$$\text{CDF} \xrightarrow{\text{discrete}} \text{PMF}$$

$$\text{CDF} \xrightarrow{\text{cont.}} \text{PDF}$$

Continuous RV (Ch 3.2)

Continuous RV and PDF

X is a continuous RV if its pdf f_X follows



CDF ← • $F_X(c) = \int_{-\infty}^{c} f_X(u)du$ for all $c \in \mathbb{R}$
- Support – $\{u: f_X(u) > 0\}$
- If $F_X(c)$ is continuous and differentiable, $f_X = F_X{}'$

    - Since there is no jump in $F_X(c)$, $P\{X = c\} = F_X(c) - F_X(c-)$

$a < X < b$

- $P\{a < X \le b\} = F_X(b) - F_X(a) = \int_a^b f_x(u)du$ $= 0$ $\neq f_x(c)$

$a \le X \le b$

- $\int_{-\infty}^{\infty} p_X(u)du$

$= F_x(\infty) - F_x(-\infty) = 1 - 0 = 1$

Exam-Safe

Why P "density" F

By definition, $f_X = F'_X$

$$f_X(u_0) = \lim_{h\to 0} \frac{F(u_0+h) - F(u_0)}{h}$$

$$+)\quad f_X(u_0) = \lim_{h\to 0} \frac{F(u_0) - F(u_0-h)}{h}$$

$$f_X(u_0) = \lim_{h\to 0} \frac{F(u_0+h) - F(u_0-h)}{2h}$$

- Let $\epsilon = 2h > 0$
- $f_X(u_0) = \lim_{h\to 0} \frac{F\left(u_0+\frac{\epsilon}{2}\right) - F\left(u_0-\frac{\epsilon}{2}\right)}{\epsilon}$
- $P\left\{u_0 - \frac{\epsilon}{2} < X < u_0 + \frac{\epsilon}{2}\right\} = \epsilon f_X(u_0) + O(\epsilon)$
- "Density of the probability"

Expectation and Variance

discrete

$\mu_X = \sum_u u P_X(u)$

- $\mu_X = E[X] = \int_{-\infty}^{\infty} u f_X(u) du$
- LOTUS still applies, $E[g(x)] = \int_{-\infty}^{\infty} g(u) f_X(u) du$
- E.g. $E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$

- $\sigma_X^2 = Var(X) = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$

Example

- $f_X(u) = \begin{cases} A(1-u^2) & if\ -1 \le u \le 1 \\ 0 & else \end{cases}$

- Find A, $P\{-0.5 < X < 1.5\}$, F_X, μ_X, σ_X^2

$= P\{-0.5 < X < 1\}$

$$\int_{-1}^{1} A(1-u^2)\,du = A\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} = A\,\frac{4}{3} = 1$$

$$A = \frac{3}{4}.$$

$$P\{-0.5 < X < 1\} = \int_{-0.5}^{1} \frac{3}{4}(1-u^2)\,du.$$

$$= \frac{3}{4}\left(u - \frac{u^3}{3}\right)\Big|_{-0.5}^{1}$$

$$\frac{3}{4}\left(u - \frac{u^3}{3} + c\right)$$

$$F_X = \int \frac{3}{4}(1-u^2) = \frac{3}{4}\left(u - \frac{u^3}{3} + c\right)$$

$$F_X(-1) = 0 \Rightarrow c = \frac{2}{3}.$$

$$\frac{3}{4}\int_{-1}^{1} u(1-u^2)\,du = \mu_X.$$

$$F_X(1)$$

$$= \frac{4}{3} \quad 1 - \frac{1}{3} + \frac{2}{3}$$

Example

- $f_X(u) = \begin{cases} A(1-u^2) & if \quad -1 \leq u \leq 1 \\ 0 & else \end{cases}$

- Find A, $P\{-0.5 < X < 1.5\}$, F_X, μ_X, σ_X^2