Last lecture

Binary Hypothesis Testing (Ch 2.11)

- Likelihood table
- Maximum likelihood (ML) decision rule $\rightarrow P(X|H)$
- Maximum A Posteriori (MAP) decision rule $\rightarrow P(X, H)$

$\hookrightarrow$ LRT $\begin{cases} \text{ML} \cdot \tau = 1 \\ \text{MAP } \tau = \frac{\pi_0}{\pi_1} \end{cases}$

Agenda

Binary Hypothesis Testing (Ch 2.11)

- Maximum A Posteriori (MAP) decision rule examples

Reliability & Union Bound(Ch 2.12.1)

- Definition
- Examples – network outage

Likelihood table to joint probability

Assume $\pi_1 = P(H_1) = 0.2, \pi_0 =$ $P(H_0) = 0.8$

- Decide on joint probability is same as posterior probability
- MAP rule = LRT rule with $\tau =$ 4

Pick higer $P(H, X)$ per column

or $\frac{P(X|H_1)}{P(X|H_0)} > \frac{\pi_0}{\pi_1} = \tau$

$\times \pi$ ($\times P(H)$)

Joint prob.

sus. circles in ultrasound

$P(X\|H)$	$X = 0$	$X = 1$	$X = 2$	$X = 3$
Tumor H_1	0	0.1	0.3	0.6
H_0	0.4	0.3	0.2	0.1

$\xrightarrow{\times \pi_1}$

$\xrightarrow{\times \pi_0}$

$P(H, X)$	$X = 0$	$X = 1$	$X = 2$	$X = 3$
H_1	0	0.02	0.06	0.12
H_0	0.32	0.24	0.16	0.08

Example

X: Draw a coin from the bag and toss it 5 times , # of H,

- Likelihood table
- Joint probability table
- Describe ML and MAP rule, compute
  - p_{false_alarm}
  - p_{miss}
  - p_e

$H_1: p = \frac{2}{3}$ coin

$H_0: p = \frac{1}{2}$ coin

Example

Likelihood

P_{miss} for ML

	$X=0$	1	2	3	4	5
H_1						
H_0						

$P_{false\ alarm}$ for ML

$\frac{1}{3}$

$\frac{1}{2}$

$H_1: p_1 = \frac{2}{3}$ coin

$H_0: p_0 = \frac{1}{2}$ coin

$$P(X=2\mid H_1) = \binom{5}{2} p_1^2 (1-p_1)^3$$

$$P(X=2\mid H_0) = \binom{5}{2} p_0^2 (1-p_0)^3$$

k $\quad$ $5-k$ $\quad$ $k \geq 3$

ML?

$$\Lambda_k(2) = \frac{\square}{\square} = \left(\frac{p_1}{p_0}\right)^2 \left(\frac{1-p_1}{1-p_0}\right)^3 = \left(\frac{4}{3}\right)^2 \left(\frac{2}{3}\right)^3 \geq 1$$

$\geq \frac{1}{4}$

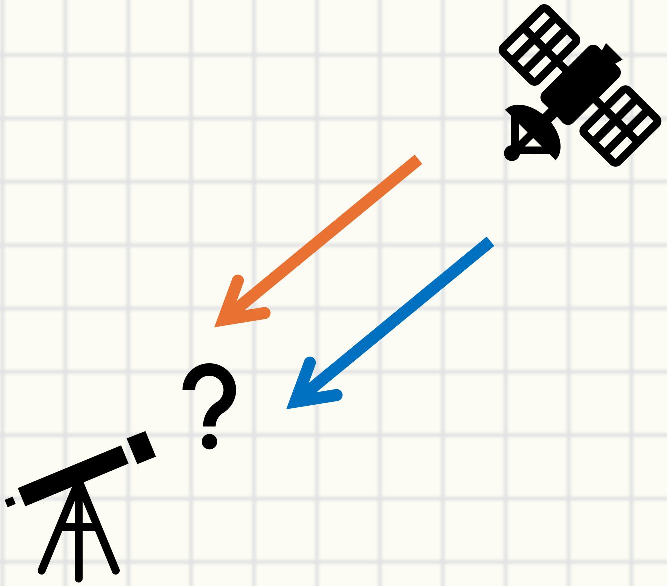
$k \geq 1$ MAP

Example

Receive on-off keying (OOK) signal from a deep space Tx.

X: # of photons observe from a telescope

- $\lambda = 6$ If it's ON
- $\lambda = 2$ If it's OFF
- $\frac{\pi_0}{\pi_1} = 5$
- Describe ML and MAP rule, compute
  - p_{false_alarm}
  - p_{miss}
  - p_e



$H_1: Pois(\lambda = 6)$

$H_0: Pois(\lambda = 2)$

Example

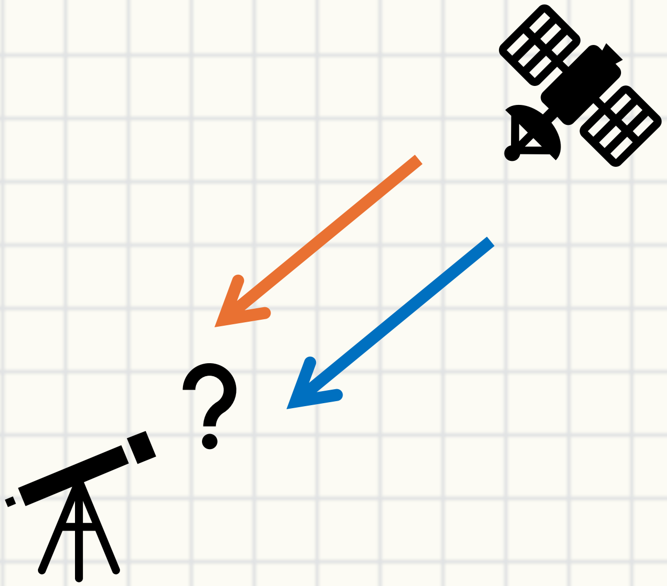
e^{-4}

$$\Lambda(k) = \frac{P(X=k|H_1)}{P(X=k|H_0)} = \frac{\frac{\cancel{e^{-6}}\,6^k}{\cancel{k!}}}{\frac{\cancel{e^{-2}}\,2^k}{\cancel{k!}}}$$

$$= e^{-4}\,3^k$$

ML ≥ 1

MAP $\geq \frac{\pi_0}{\pi_1} = 5$



$H_1: Pois(\lambda = 6)$

$H_0: Pois(\lambda = 2)$

k	1	2	3	4
e^{-k}				

Reliability / Union Bound

Motivation and Definition

Reliability

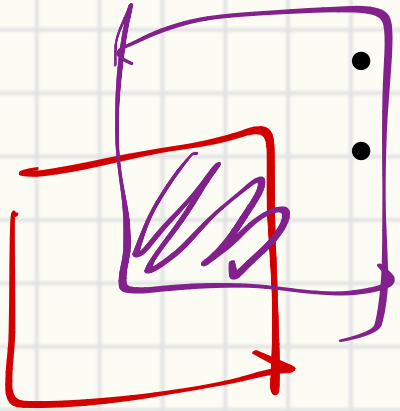
- How likely a system will fail?
  - Each subsystem fail with small probability independently
  - If sub-systems fail in some pattern, the system fails

↳ # of subsystem

comb. subsystem

Union Bound

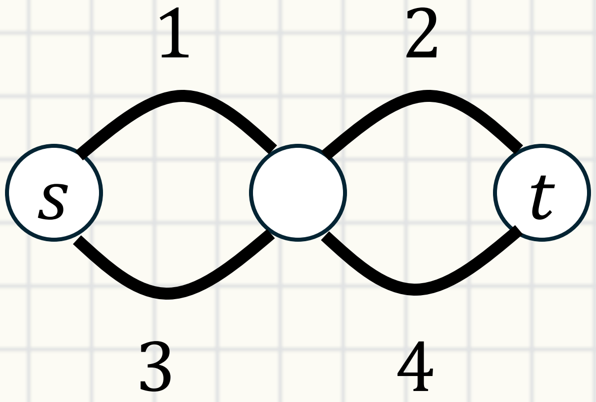
- Bounds for union of small probability events $P(A)+P(B) - P(A\cap B)$
- $P(A \cup B) \le P(A)+P(B)$
- $P(A_1 \cup A_2 \cup \cdots \cup A_m) \le \sum_m P(A_m)$
- Bound is at most 2x far from the actual value $P(A\cap B) \le \min\{P(A), P(B)\}$

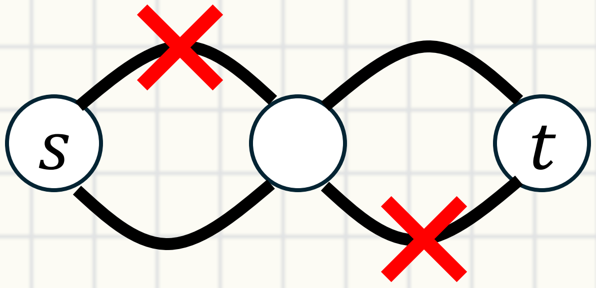


Example – Network outage

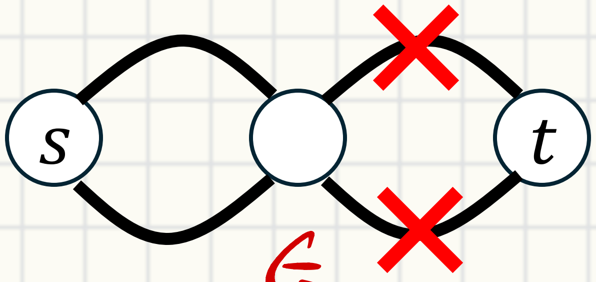
A $s - t$ network consists of nodes source (s), terminal (t), other nodes, and links.

- Each link k fails independently with small probability p_k
- Network outage event F occurs if
  - For any path from s to t, there is at least one link in the path fails



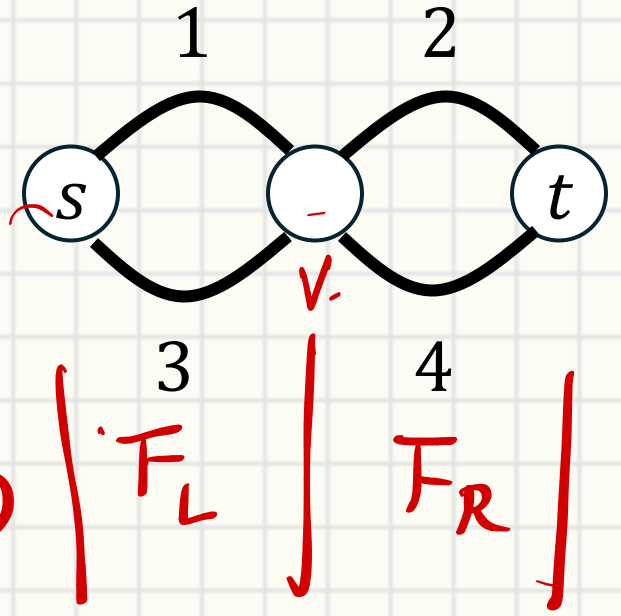


$x_1 \notin F$



$x_2 \in F$

Example – Network outage



Compute $P(F)$

- $P(F) = P(F_L \cup F_R)$

$= P(F_L) + P(F_R) - P(\underline{F_L \cap F_R})$

$= P_1P_3 + P_2P_4 - P_1P_2P_3P_4$

Exact probability $p_k = 0.001$

$P(F)$

$= 10^{-6} + 10^{-6} - \underline{10^{-12}}$

$\approx 2 \times 10^{-6}$

Union bound $p_k = 0.001$

$P(\underset{F_L}{A} \cup \underset{F_R}{B}) \leq P(\underset{F_L}{A}) + P(\underset{F_R}{B})$

$= 10^{-6} + 10^{-6}$

$= 2 \times 10^{-6}$