Last lecture

Independent Events/ RVs (Ch 2.4)

- Examples and Facts

Distributions (Ch 2.4)

- Bernoulli
- Binomial

$$p_X(1) = p$$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Agenda

Example for binomial distribution

Geometric distribution (Ch 2.5)

- Definition
- Examples
- Property – memoryless

Bernoulli Process (Ch 2.6)

- Definition
- Properties
- Negative binomial distribution

Binomial Example – Best of K

$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

→ odd

Team A and B play "Best of 7" games

- No tie, whoever wins 4 games out of 7 is the match winner
- E.g. $w_i = \{A, A, A, B, A\}$: the winner is A ← 4th victory
- Let p denotes A's win rate per game
- $\boxed{Y}$ denotes the number of games played, $p_Y(k) = ?$

$Y_i = 5$

$$P_Y(k) = P_{Y|A}(k) + P_{Y|B}(k)$$

↑ A wins match ↑ B wins.

A wins kth game

$$P_{Y|A}(k) = \binom{k-1}{3} P^{3} (1-P)^{k-1-3} \cdot P$$

$k-1-3$

B

$\underbrace{\qquad\qquad}_{}$ 3 As

_ _ _ _ _ A
k

$$f)\ \frac{P_{Y|B}(k) = \binom{k-1}{3}(1-p)^3\, p^{k-1-3}(1-p)}{P_Y(k)}$$

Geometric Distributions

Geometric Distribution

of Toss on a (unfair) coin until the first Head is shown

p

Conduct independent Bernoulli trials of parameter p

- $L \triangleq$ # of trials until we get the first 1
- $p_L(1) = p$ (H)
- $p_L(2) = \underbrace{(1-p)}_{\text{1st Tail}} \times \underbrace{p}_{\text{Head}}$ (T, H)
- $p_L(k) = (1-p)^{k-1} \times p$ $(\overbrace{T, T, T \ldots}^{k-1} H)$
- $P\{L > k\} = (1-p)^k$

$\rightarrow (\overbrace{T, T, T \ldots T}^{k}, ?, ?, ?)$

Mean

1. $\mu_L = \sum_{k=1}^{\infty} \underbrace{(1-p)^{k-1} p}_{P_L(k)\cdot k} k$

$$E[L] = 1 \times p + (1-p) \times (E[L] + 1)$$

$\overbrace{T \quad T\ T \ \cdots\ T\ H}^{k}$

$\underbrace{\#1 \quad \#2 \sim \text{End}}_{L_2 = k}$

$L \begin{cases} k+1 & \text{if } X_1 = T \\ 1 & \text{if } X_1 = H \end{cases}$

Recursive. $E[L_2] = E[L]$

$E[L] = P\{X_1 = H\} \times 1 + P\{X_1 = T\} \times E[L_2 + 1]$

$E[L] = \frac{1}{p}$

Variance

$$Var(L) = \underline{E[L^2]} - \underline{\mu_L^2}$$

X_1 X_2 $\rightarrow$

H

T

$$Var(L) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$

$$E[L] = \frac{1}{p}$$

$$E[L^2] = \underbrace{p \times (1)^2}_{X_1 = H} + \underbrace{(1-p) \times E[(L+1)^2]}_{X_1 = T}$$

$$= p + (1-p)\, E[L^2 + 2L + 1]$$

$$= (\cancel{1}-p)\, E[L^2] + \frac{2(1-p)}{p} + (1-\cancel{p}) + \cancel{p}$$

$$p\, E[L^2] = \frac{2-p}{p} \qquad E[L^2] = \frac{2-p}{p^2}$$

Example

What's the expected number of rolls to get 1 to 6 at least once?

- Example series :
  $\{2, 4, 2, 3, 4, 4, 3, 5, 3, 5, 4, 4, 6, 2, 3, 3, 4, 1\}$

  $R_1 \quad R_2 \quad R_3 \quad R_4 \quad R_5 \quad R_6$

- $R_k \triangleq$ # roll to get the k-th unseen number

Notation

$R_k \sim G(P_k)$

$$E[R] = \sum_{k=1}^{6} E[R_k]$$

$$P_1 = \frac{|\text{Unseen}|}{|\Omega|} = \frac{6}{6}$$

$R_1 \sim G(1)$

$R_2 \sim G(\frac{5}{6})$ $P_2 = \frac{|\text{Unseen}|}{|\Omega|} = \frac{5}{6}$

$R_3 \sim G(\frac{4}{6})$ $P_3 = \frac{|\text{Unseen}|}{|\Omega|} = \frac{4}{6}$

$E[R] = E[G(1)] + [G(\frac{5}{6})] \dots G[\frac{1}{6}]$

$= 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 6$

$E[G] = \frac{1}{P}$

Property – Memoryless property

For geometric ~~series~~ distribution, failing 10 times will not affect the 11-th trial

- $P\{L > k + n | L > n\} =$ $P\{L > k\}$

- Called "memoryless property"
- What's the expected total number to get the first 1 after getting {0,0,0,0}? $= \frac{1}{p} + 4$