

# Last lecture

## Independent Events/ RVs (Ch 2.4)

- Examples and Facts

## Distributions (Ch 2.4)

- Bernoulli
- Binomial

$$p_X(1) = p$$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

# Agenda

Example for binomial distribution

Geometric distribution (Ch 2.5)

- Definition
- Examples
- Property – memoryless

Bernoulli Process (Ch 2.6)

- Definition
- Properties
- Negative binomial distribution

# Binomial Example – Best of K

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

→ odd

Team A and B play “Best of 7” games

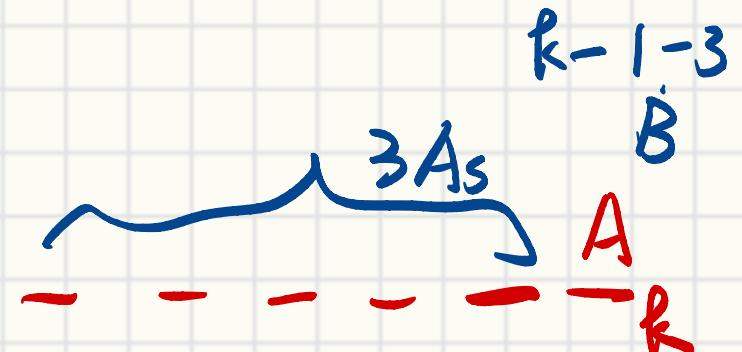
- No tie, whoever wins 4 games out of 7 is the match winner
- E.g.  $w_i = \{A, A, A, B, A\}$ : the winner is A 4<sup>th</sup> victory
- Let  $p$  denotes A's win rate per game
- $\boxed{Y}$  denotes the number of games played,  $p_Y(k) = ?$

$Y_i=5$

$$P_Y(k) = P_{Y|A}(k) + P_{Y|B}(k)$$

$\sum$  A wins match      B wins.

A wins k-th  
game



$$P_{Y|A}(k) = \binom{k-1}{3} P^3 (1-P)^{k-1-3} \cdot P$$

$$+) P_{Y|B}(k) = \binom{k-1}{3} (1-p)^3 p^{k-1-3} (1-p)$$

$$\overbrace{P_Y(k)}$$

## Geometric Distributions

# Geometric Distribution

# of Toss on a (unfair) coin until the first Head is shown  
P.

Conduct independent Bernoulli trials of parameter  $p$

- $L \triangleq \# \text{ of trials until we get the first } 1$

- $\underline{p_L(1)} = P.$

- $\underline{p_L(2)} = \underbrace{(1-P)}_{\text{1st Tail}} \times \underbrace{P}_{\text{Head}}$

- $\underline{p_L(k)} = (1-P)^{k-1} \times P.$

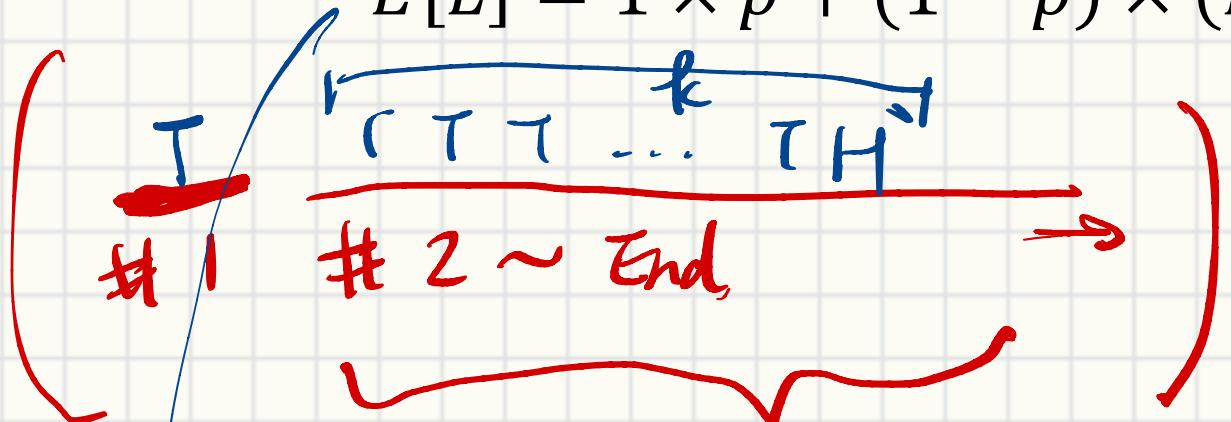
- $\underline{P\{L > k\}} = (1-P)^k$

$\hookrightarrow (\overbrace{T, T, T, \dots, T}^k, ?, ?, ?, ?)$

$\underbrace{(T, T, T, \dots, H)}_{k-1}$

# Mean

$$E[L] = 1 \times p + (1 - p) \times (E[L] + 1)$$



$$L_i = \begin{cases} k+1 & \text{if } X_i = T, \\ 1 & \text{if } X_i = H. \end{cases}$$

Recursive.

$$\downarrow$$

$$E[L_2] = E[L]$$

$$E[L] = P\{X_1 = H\} \times 1 + P\{X_1 = T\} \times E[L_2 + 1]$$

$$E[L] = \frac{1}{P}$$

$$\textcircled{1} \quad \mu_L = \sum_{k=1}^{\infty} \frac{(1-p)^{k-1} p^k k}{P_L(k) \cdot k}$$

# Variance

$\frac{x_1}{x_1} \quad x_2 \longrightarrow$

$$\underline{Var(L)} = E[L^2] - \underline{\mu_L^2}$$

$$\frac{H}{T}$$

$$\cancel{E[L^2]} = p \times (1)^2 + (1-p) \times E[(L+1)^2]$$

$X_1 = H$        $X_1 = T$

$$\begin{aligned} Var(L) &= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\ &= \frac{1-p}{p^2} \end{aligned}$$

$$E[L] = \frac{1}{p}$$

$$= p + (1-p) E[L^2 + \cancel{2L} + 1]$$

$$= (1-p) E[L^2] + \frac{2(1-p)}{p} + (1-p) + p$$

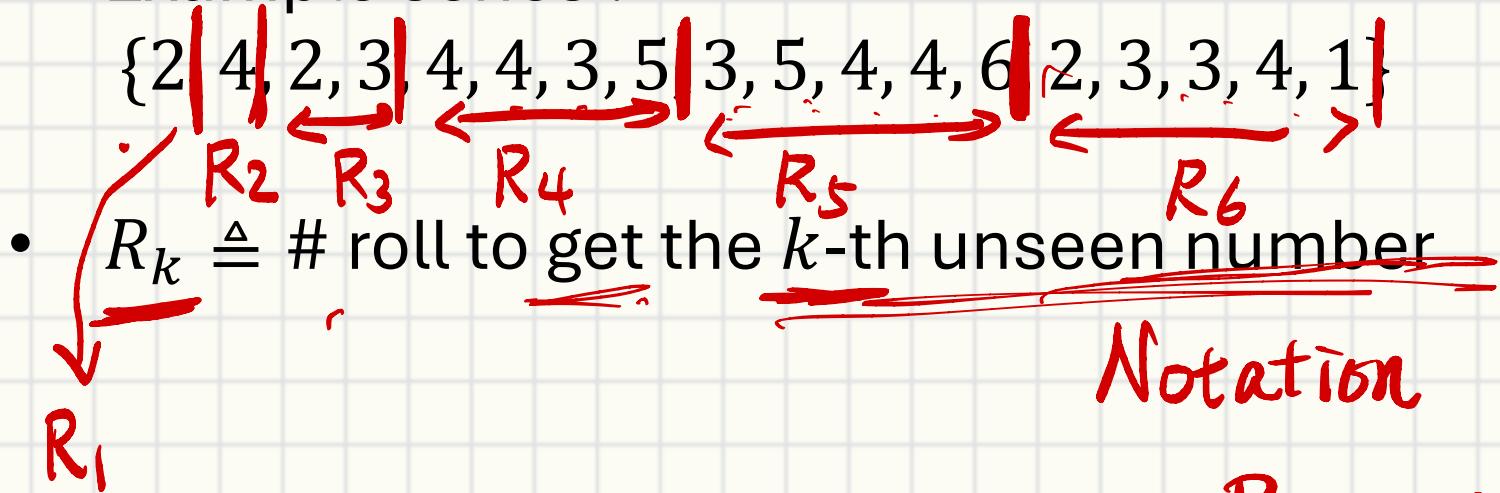
$$P E[L^2] = \frac{2-p}{p}$$

$$E[L^2] = \frac{2-p}{p^2}$$

# Example

What's the expected number of rolls to get 1 to 6 at least once?

- Example series :



- $R_k \triangleq \# \text{ roll to get the } k\text{-th unseen number}$

Notation

$$E[R] = \sum_{k=1}^6 E[R_k]$$

$$R_k \sim G(p_k)$$
$$p_1 = \frac{|\text{Unseen}|}{|\Sigma|} = \frac{6}{6}$$

$$R_1 \sim G(1)$$

$$R_2 \sim G\left(\frac{5}{6}\right) P_2 = \frac{\text{Unseen}}{|\Sigma|} = \frac{5}{6}$$

$$R_3 \sim G\left(\frac{4}{6}\right) P_3 = \frac{\text{Unseen}}{|\Sigma|} = \frac{4}{6}$$

$$\begin{aligned} E[R] &= E[\underbrace{G(1)}_{\text{...}}] + [G\left(\frac{5}{6}\right)] \dots G\left(\frac{1}{6}\right) \\ &= 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 6 \end{aligned}$$

$$E[G] = \frac{1}{P_r}$$

# Property – Memoryless property

For geometric ~~series~~, failing 10 times will not affect the 11-th trial  
*distribution*

- $P\{L > k + n | L > n\} = P\{L > k\}$
- Called “memoryless property”
- What’s the expected total number to get the first 1 after getting {0,0,0,0}?

$$-\frac{1}{P_1} + 4$$