Last lecture

Bayes formula (Ch 2.10)

- Examples

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Independent Events/ RVs (Ch 2.4)

- Definition
- Motivation
- Examples and Facts

$P(B|A) = P(B)$

$P(AB) = P(A)P(B)$

Agenda

Independent Events/ RVs (Ch 2.4)

- Examples and Facts

Distributions (Ch 2.4)

- Bernoulli
- Binomial

Geometric distribution (Ch 2.5)

- Definition

Terms and Facts

- A, B, C are pairwise independent if $(A, B), (B, C), (A, C)$ are mutually independent
  - Toss a fair coin twice
  - $A \triangleq$ {First coin is Head}
  - $B \triangleq$ {second coin is Head}
  - $C \triangleq$ {toss results are the same}

Does not imply

A, B, C are all independent

$P(ABC) \overset{?}{=} P(A)P(B)P(C)$

$\Omega_A = \{(H,T), (H,H)\}$ $\quad \|\Omega\| = 4 \quad P(A) = P(B) = P(C) = \frac{1}{2}$

$\Omega_B = \{(T,H), (H,H)\}$

$\Omega_C = \{(H,H), (T,T)\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = P(A).$$

$$P(B|C) = \frac{1}{2}, \quad P(A|C) = \frac{1}{2}.$$

$$P(ABC) = \frac{1}{4} \neq P(A)P(B)P(C)$$

Terms and Facts

$F(B,C)$ if

- A, B, C are independent they are pairwise independent and $P(ABC) = P(A)P(B)P(C)$

Proof $P(A \cap (B \cup C))$

$= P(A)\,P(B \cup C)$

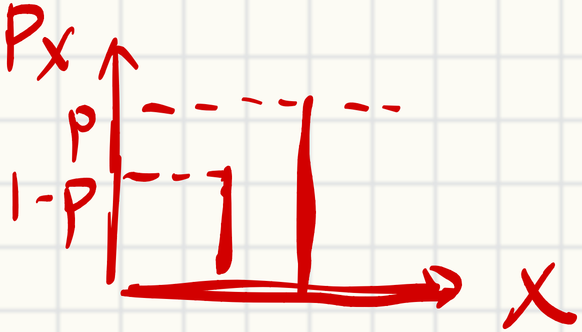
- $A_1, A_2, \ldots A_i$ are independent if
$$P\left(A_{i_1}A_{i_2}\ldots A_{i_k}\right) = P\left(A_{i_1}\right)P\left(A_{i_2}\right)\ldots P\left(A_{i_k}\right)$$

Distributions

$\Downarrow$

$\{X, P_X\}$ of commonly seen exp.

Bernoulli Distribution

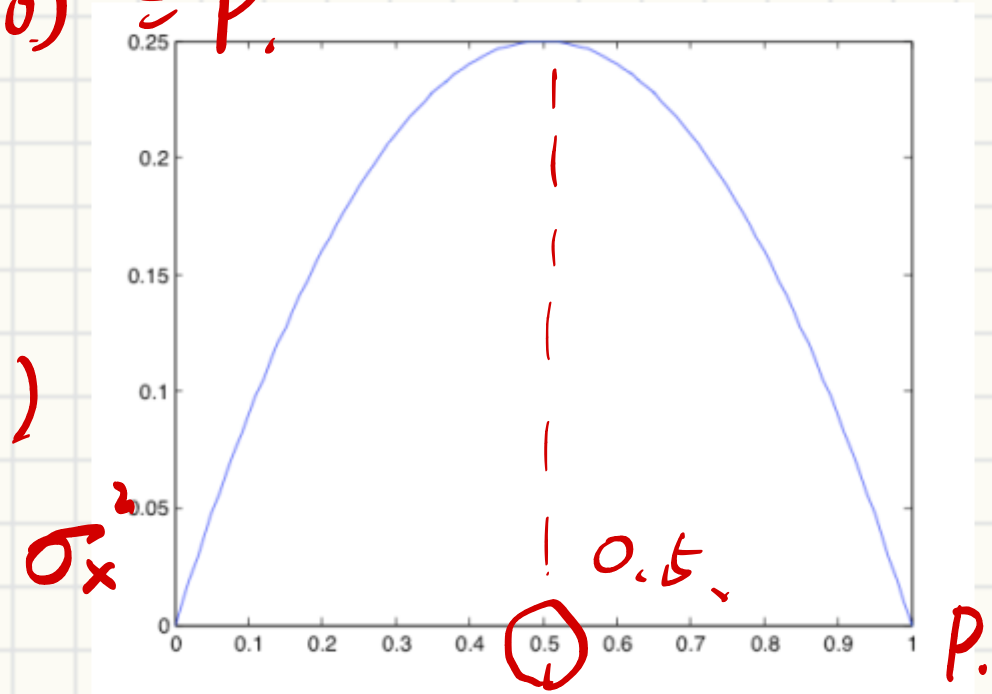


X is Bernoulli distribution with parameter p if

- $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$
- "Toss a (unfair) coin with p probability Head"
- Only 2 possible outcomes, pmf contains 2 bins
- $E[X] = 1 \times P_X(1) + 0 \cdot P_X(0) = P.$
- $E[X^2] = P$
- $\sigma_x^2 =$ $E[X^2] - (E[X])^2$ $= P - P^2 = P(1-P)$

first momentum

second momentum



Binomial Distribution

X is binomial distribution with parameter (n, p) if $\Rightarrow n+1$ bins $[0, n]$

- X is sum of n Bernoulli trials with parameter p
- Draw the unfair coin n times and count the Head

$(n, p) = (24, \frac{1}{3})$

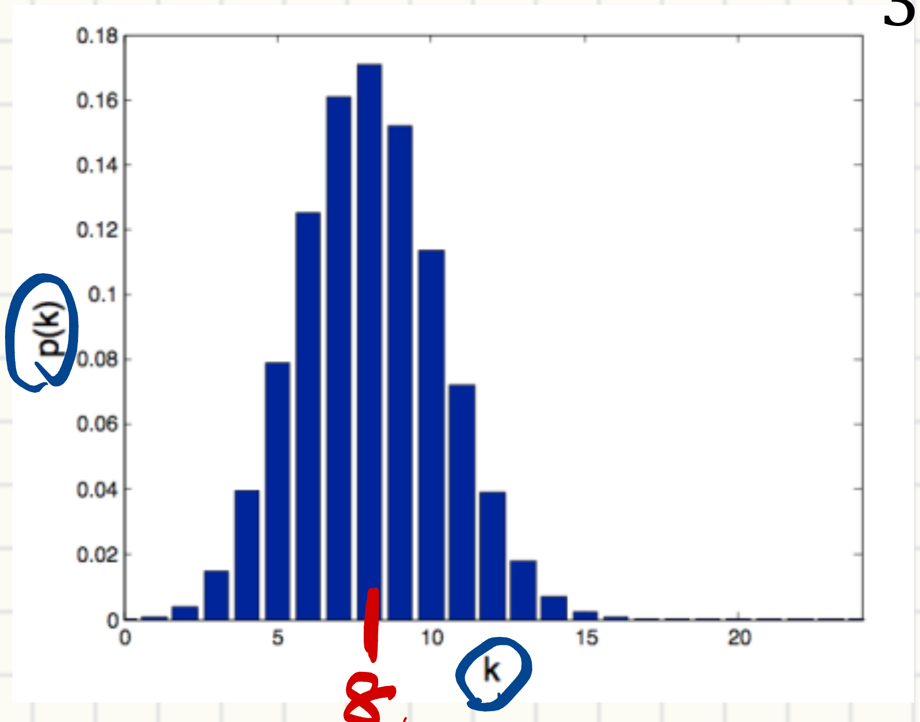
- $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Head

$P_X(2)$ for $(n, p) = (4, 0.5)$

$\binom{4}{2}$ HHTT, HTHT, HTTH, TTHH, THTH, THHT.

$\underbrace{\frac{1}{2} \times \frac{1}{2}}_{p^2} \times \underbrace{\frac{1}{2} \times \frac{1}{2}}_{(1-p)^{4-2}}$



Binomial Distribution

$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\boxed{\text{Exam-safe}}$

Proof - $\sum_{k=0}^{n} p_X(k) = 1$

- $(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k}$
- $x = \frac{p}{1-p}$

$x^n + \binom{n}{1} x^{n-1} \ldots$

$\left(1+\frac{p}{1-p}\right)^n = \sum \binom{n}{k} \left(\frac{p}{1-p}\right)^k$

$1 = \sum \binom{n}{k} p^k (1-p)^{n-k}$

$\left(\frac{1}{1-p}\right)^n = \sum \binom{n}{k} p^k \left(\frac{1}{1-p}\right)^k$

$\times (1-p)^n$

Binomial mean

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$\boxed{\text{Exam-Safe}}$

$$E[X] = \sum_{k=0}^{n} k \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{p_X(k)}$$

$$= \sum_{k=1}^{n} k \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{m=0}^{n-1} \frac{n-1!}{(n-1-m)!m!} p^m (1-p)^{n-1-m}$$

$$= np \sum_{m=0}^{j} \binom{j}{m} p^m (1-p)^{j-m} = np.$$

Binomial Properties

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Mean $E[X] =$ np
- Variance $Var(X) =$ $np(1-p) = n \times Var(\text{Bernoulli}(p))$
  - why?
    n tosses are independent.

- Shape of the pmf- what is the most likely k?
  - $k^* = \lfloor (n+1)p \rfloor$

$$Var(X_1+X_2) = Var(X_1) + Var(X_2)$$

$\downarrow$ if X_1, X_2 are independent

$P_{X_1 X_2}(j,k)$

$$E[(X_1+X_2)^2] - (E[(X_1+X_2)])^2$$

$$= E[X_1^2] + E[2X_1X_2] + E[X_2^2] - (\mu_1+\mu_2)^2$$

$$= E[X_1^2] - \mu_1^2 + E[X_2^2] - \mu_2^2 + E[2X_1X_2] - 2\mu_1\mu_2$$

$$= Var(X_1) + Var(X_2)$$

independ

$$\sum_j \sum_k jk \, P_{X_1}(j) P_{X_2}(k)$$

μ_1 μ_2

$$= \left[\sum_j j P_{X_1}(j)\right]\left[\sum_k k P_{X_2}(k)\right]$$

Slido

#5018375

Back to early 1900's, there's a mail fraud

- I mail N people that I can predict a series of 50-50 games
- I predict A wins in $\frac{N}{2}$ mail, B wins for the other
- Stop mailing a person after 2 wrong guess
- Say people will subscribe to my prediction if I only make at most 1 mistake in 5 guesses
- What should be N if I want to get 300 subs?

4 correct 5 correct

1600

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P_X(5) = \binom{5}{5}\left(\frac{1}{2}\right)^5 \cancel{\left(\frac{1}{2}\right)^{5-5}} \rightarrow 1$$

$$= \frac{1}{32}$$

$$+)\quad P_X(4) = \binom{5}{4}\left(\frac{1}{2}\right)^4 \underline{\left(\frac{1}{2}\right)^{5-4}}$$

$$= 5 \times \frac{1}{32}$$

$$\frac{6}{32} = \frac{3}{16}$$