ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 9: CONDITIONAL PROBABILITIES

• TOPICS TO COVER (BASED ON CH 2.3)

→ CONDITIONAL PROBABILITIES

• https://montyhall.io/

• RE ⇝ $(\Omega, \mathcal{F}, P)$ $\quad P: \mathcal{F} \longrightarrow \mathbb{R}$

ASSUME EVENT A ALREADY OCCURRED!

$$\Omega \longrightarrow \Omega \cap A := \Omega_A$$

$$\mathcal{F} \longrightarrow \mathcal{F} \cap A := \{ F \cap A : F \in \mathcal{F} \} := \mathcal{F}_A$$

$$P(B) \longrightarrow \frac{P(B \cap A)}{P(\Omega \cap A)} \qquad B \in \mathcal{F}$$

$$P(B) = \frac{P(B \cap \Omega)}{P(\Omega)}$$

$$\frac{P(B \cap A)}{P(A)} : \text{CONDITIONAL PROB. OF B GIVEN A}$$

$\because S = S \cap \Omega \quad \forall S \subset \Omega$

$:= P(B|A) \quad$ if $P(A) > 0$

UNDEFINED $:= P(B|A) \quad$ if $P(A) = 0$

$(\Omega, \mathcal{F}, P) \xrightarrow{\text{EVENT A OCCURRED}} (\Omega_A, \mathcal{F}_A, P(\cdot|A))$

ORIGINAL PROB. SPACE $\qquad$ CONDITIONAL PROB. SPACE

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \qquad P(A) > 0$$

DEFINITION: INDEPENDENCE OF EVENTS

IF $P(B) = P(B|A)$: B AND A ARE INDEPENDENT

PROPERTIES OF CONDITIONAL PROB. MEASURE :

$P(A) > 0$ and B and C are some events.

1. $P(B|A) \geq 0$ HOLDS BY DEF.

2. $P(B|A) + P(B^c|A) = 1$

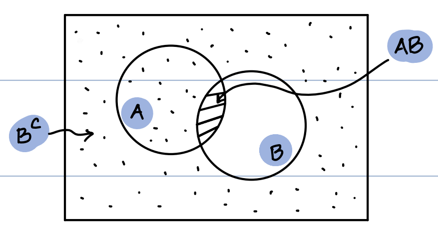
FOR THE 'USUAL' PROB. MEASURE: $P(B) + P(B^c) = 1$

$= P(B|\Omega) + P(B^c|\Omega) = 1$

IN GENERAL, $E_1, E_2, \dots$ ARE DISJOINT $\Rightarrow P(E_1 \cup E_2 \cup \dots | B) = P(E_1|B) + P(E_2|B) + \dots$

PROOF:

$$P(B|A) \overset{\text{DEF.}}{=} \frac{P(AB)}{P(A)} \qquad P(B^c|A) \overset{\text{DEF.}}{=} \frac{P(AB^c)}{P(A)}$$



B AND B^c ARE DISJOINT (BY DEF.)

$\Rightarrow$ AB AND AB^c ARE ALSO DISJOINT

$\because A \cap B \subset A$ and $A \cap B \subset B$ $\forall A, B$

$$P(B|A) + P(B^c|A) = \underbrace{\frac{P(AB)}{P(A)} + \frac{P(AB^c)}{P(A)}}_{\text{DISJOINT}}$$

$$= \frac{P(AB \cup AB^c)}{P(A)} \qquad \text{AXIOM P.2}$$

$$= \frac{P(A)}{P(A)} = 1 \qquad \because AB \cup AB^c = A$$

3. $P(\Omega \mid A) = 1$

PROOF: $P(\Omega \mid A) \overset{\text{DEF.}}{=} \dfrac{P(\Omega \cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

4. $P(AB) = P(A)\, P(B \mid A)$

PROOF: $P(B \mid A) \overset{\text{DEF.}}{=} \dfrac{P(AB)}{P(A)} \Rightarrow P(AB) = P(A) \cdot P(B \mid A)$

5. $P(ABC) = P(C)\, P(B \mid C)\, P(A \mid BC)$ ASSUMING $P(BC) > 0$

$$P(A \mid BC) \overset{\text{DEF.}}{=} \frac{P(ABC)}{P(BC)} \underset{\text{PROPERTY 4}}{=} \frac{P(ABC)}{P(C)\, P(B \mid C)}$$

$\Rightarrow P(ABC) = P(C)\, P(B \mid C)\, P(A \mid BC)$

PROPERTIES 1 TO 3 $\Rightarrow$ $P(\cdot \mid A)$ IS ALSO A PROB. MEASURE.