ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 6 : RANDOM VARIABLES AND PROBABILITY MASS FUNCTIONS

- TOPICS TO COVER (BASED ON CH 2.1)

  → DISCRETE-TYPE RANDOM VARIABLES

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A RV X is said to be discrete-type if there is a finite set $u_1, u_2, \ldots, u_n$ or countably infinite set $u_1, u_2, \ldots,$ such that

$$P(\{X \in \{u_1, u_2, \ldots\}\}) = 1$$

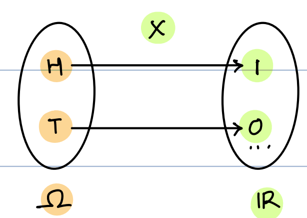
- E.G. TOSS A COIN

↳ $(\{H, T\}, \{\Phi, \{H\}, \{T\}, \Omega\}, P_1)$

$P_1 : \mathcal{F} \longrightarrow \mathbb{R}$

$P_1(\Phi) = 0, \quad P_1(\Omega) = 1$

$P_1\{H\} = \frac{1}{2}, \quad P\{T\} = \frac{1}{2}$



$P\{X \in \{0\}\} = P\{\omega : X(\omega) \in \{0\}\} = P\{T\} = \frac{1}{2}$

$P\{X \in \{0, 1\}\} = P\{\omega : X(\omega) \in \{0, 1\}\} = P\{H, T\} = 1$

X is a discrete-type RV as $P\{X \in \underbrace{\{0, 1\}}_{\text{finite set}}\} = 1$

• THE PROB. MASS FUNCTION (PMF) OF A DISCRETE-TYPE RV X:

$$p_X(u) := P\{X = u\} \geq 0$$

such that

$$\sum_{i} p_X(u_i) = 1$$

$i = 1, 2, \ldots$

We can determine the probability of any event determined by X using its PMF:

$$P(\{X \in B\}) := \sum_{i : u_i \in B} p_X(u_i)$$

• THE SUPPORT OF A PMF IS THE SET OF u SUCH THAT

$$p_X(u) > 0$$

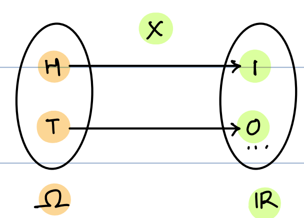
• E.G. TOSS A COIN

$\hookrightarrow (\{H, T\}, \{\phi, \{H\}, \{T\}, \Omega\}, P_1)$

$P_1 : \mathcal{F} \longrightarrow \mathbb{R}$

$P_1(\phi) = 0, \quad P_1(\Omega) = 1$

$P_1\{H\} = \frac{1}{2}, \quad P\{T\} = \frac{1}{2}$



$$P\{X \in \{0\}\} = P\{\omega : X(\omega) \in \{0\}\} = P\{T\} = \frac{1}{2}$$

$$P\{X \in \{0, 1\}\} = P\{\omega : X(\omega) \in \{0, 1\}\} = P\{H, T\} = 1$$

• PMF OF X: $p_X(0) = \frac{1}{2} = p_X(1)$ NOTE THAT: $p_X(0) + p_X(1) = 1$

• SUPPORT OF THE PMF OF X: $\{0, 1\}$ AS $p_X(u) > 0, \; u = 0, 1$

- E.G. ROLL A DIE

X = FACE YOU GET WHEN YOU ROLL A 6-FACED DIE

- PMF OF X : $p_X(u) = P\{X = u\}$

X	$p_X(u) = P\{X = u\}$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

- SUPPORT : $\{1, 2, 3, 4, 5, 6\}$

- E.G. ROLLING TWO DICE TOGETHER

Y_1 = SUM OF THE FACE ON EACH DIE

X_1 = FACE OF DIE 1 $\quad X_2$ = FACE OF DIE 2

RANGE OF Y_1 : $Y_1 = X_1 + X_2$; $2 \leq Y_1 \leq 12$

Y_1	(X_1, X_2)	Y_1	(X_1, X_2)
2	(1, 1)	5	(1, 4), (2, 3), (3, 2), (4, 1)
3	(1, 2), (2, 1)	6	
4	(1, 3), (2, 2), (3, 1)	7	

Y_1	(X_1, X_2)	Y_1	(X_1, X_2)
8		11	$(5,6)$, $(6,5)$
9		12	$(6,6)$
10			

TOTAL POSSIBILITIES ARE : $6 \cdot 6 = 36$

• PMF OF Y_1

$Y_1 = u$	$p_{Y_1}(u)$	$Y_1 = u$	$p_{Y_1}(u)$
2	$1/36$	7	
3	$2/36$	8	
4	$3/36$	9	
5	$4/36$	10	
6	$5/36$	11	$2/36$
		12	$1/36$

IN GENERAL:

$$P(Y_1 = u) = \begin{cases} \dfrac{u-1}{36}, & \text{if } u = 2, 3, \dots, 7, \\ \dfrac{13-u}{36}, & \text{otherwise}. \end{cases}$$

NOTE THAT:

$$\sum_{u_i} p_{Y_1}(u_i) = \sum_{u=2}^{7} \frac{u-1}{36} + \sum_{u=8}^{12} \frac{13-u}{36}$$

$$= \frac{1}{36}\left(\sum_{u=2}^{7} u - \sum_{u=2}^{7} 1\right) + \frac{1}{36}\left(\sum_{u=8}^{12} 13 - \sum_{u=8}^{12} u\right)$$

$$= \frac{1}{36}\left(2+3+4+5+6+7 - (7-2+1)\right) + \frac{1}{36}\left(13(12-8+1) - 8+9+10+11+12\right)$$

$$= \frac{1}{36}\left(27 - 6 + 65 - 50\right) = \frac{36}{36} = 1$$

- $Y_2 = \max(X_1, X_2)$

- PMF OF Y_2 : $p_{Y_2}(u) = P\{Y_2 = u\} = P\{\max(X_1, X_2) = u\}$

RANGE OF Y_2 : $Y_2 = \max(X_1, X_2)$ $\quad 1 \le Y_2 \le 6$

$Y_2 = u$	(X_1, X_2)	$p_{Y_2}(u)$
1	(1, 1)	1/36
2	(1, 2), (2, 2), (2, 1)	3/36
3		
4		
5		
6		

$$p_{Y_2}(u) = P\{Y_2 = u\} = P\{\max(X_1, X_2) = u\}$$

CONSIDER : $P\{Y_2 = u\} = P\{Y_2 \le u\} - P\{Y_2 \le u-1\}$ ... (1)

$$P\{Y_2 \le u\} = P\{\max(X_1, X_2) \le u\}$$

$$= P\{X_1 \le u, \ X_2 \le u\}$$

INDEPENDENCE OF EVENTS

$$= P\{X_1 \le u\} \cdot P\{X_2 \le u\}$$

$$= \frac{u}{6} \cdot \frac{u}{6} = \frac{u^2}{36}$$

SIMILARLY $P\{Y_2 \le u-1\} = \frac{(u-1)^2}{36}$

(i) $\Rightarrow$ $P\{Y_2 = u\} = \dfrac{u^2}{36} - \dfrac{(u-1)^2}{36} = \dfrac{2u-1}{36}$ $u = 1, 2, 3, 4, 5, 6$

PMF OF Y_2

SUPPORT OF THE PMF

NOTE THAT: $\sum_{u_i} p_{Y_2}(u_i) = \sum_{u=1}^{6} \dfrac{2u-1}{36} = \dfrac{1}{36}\left\{ 2\sum_{u=1}^{6} u - \sum_{u=1}^{6} 1 \right\}$ $\because \sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

$$= \frac{1}{36}\left\{ 2 \cdot \frac{6 \cdot 7}{2} - 6 \right\}$$

$$= \frac{1}{36}\left\{ 42 - 6 \right\} = 1$$