ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 38 : JOINT GAUSSIAN DISTRIBUTION

- TOPICS TO COVER (BASED ON CH 4.11)

  → JOINT GAUSSIAN DISTRIBUTION

→ JOINT GAUSSIAN DISTRIBUTION

$\binom{X}{Y}$: A RANDOM VECTOR

DEFINITION. RANDOM VARIABLES $\underbrace{X \text{ AND } Y}$ ARE SAID TO BE JOINTLY GAUSSIAN IF EVERY LINEAR COMBINATION $\overbrace{aX + bY}^{\text{A NEW RV}}$ ← UNIVARIATE, $a, b \in \mathbb{R}$, IS A $\overset{\text{UNIVARIATE}}{\text{GAUSSIAN}}$ RV. FOR THE PURPOSE OF THIS DEFINITION, A CONSTANT IS CONSIDERED TO BE A GAUSSIAN RV WITH VARIANCE ZERO.

$a = 0$ AND $b = 0$ IN $aX + bY$

BEING JOINTLY GAUSSIAN INCLUDES THE CASE THAT X AND Y ARE GAUSSIAN AND 'PERFECTLY' LINEARLY RELATED: $\underbrace{X = a_1 Y + b_1}$ OR $\underbrace{Y = a_2 X + b_2}$ FOR SOME a_1, b_1, a_2, b_2. IN THESE CASES X AND Y DO NOT HAVE A JOINT PDF.

$\rho_{X,Y} = \pm 1$

ASIDE FROM THESE TWO DEGENERATE CASES, A PAIR OF JOINTLY GAUSSIAN RVS HAS A BIVARIATE NORMAL (OR GAUSSIAN) PDF, GIVEN BY

$$f_{X,Y}(u,v) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2(1-\rho^2)}\right) \quad \cdots (*)$$

$u, v \in \mathbb{R}$; $\mu_X, \mu_Y \in \mathbb{R}$; $\sigma_X, \sigma_Y > 0$; $|\rho| < 1$

PART OF THE PROPOSITION ON PAGE 3

WE WILL SHOW IT SOON BUT

$$E(X) = \mu_X, \quad Var(X) = \sigma_X^2, \quad E(Y) = \mu_Y, \quad Var(Y) = \sigma_Y^2, \quad \rho_{X,Y} = \rho$$

ZERO MEANS, UNIT VARIANCES, AND NO CORRELATION

→ FROM STANDARD 2-D NORMAL TO GENERAL 2-D NORMAL (*)

JOINT FACTORIZES INTO ITS MARGINALS

N(0,1)

SUPPOSE W AND Z ARE INDEPENDENT, STANDARD NORMAL RVS. THEIR JOINT PDF IS THE PRODUCT OF THEIR INDIVIDUAL PDFS:

$$f_{W,Z}(\alpha, \beta) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2+\beta^2}{2}}$$

$\alpha, \beta \in \mathbb{R}$

: STANDARD BIVARIATE NORMAL PDF

: SPECIAL CASE OF THE GENERAL PDF (*) WITH ZERO MEANS, UNIT VARIANCES, AND NO CORRELATION

HOW? : LECTURE 35

THE GENERAL BIVARIATE PDF (4) CAN BE OBTAINED FROM $f_{W,Z}$ BY A LINEAR TRANSFORMATION. SPECIFICALLY, IF X AND Y ARE RELATED TO W AND Z BY

CHANGE OF SCALE

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \underbrace{\begin{pmatrix} W \\ Z \end{pmatrix}}_{\text{2-D STD NORMAL}} + \underbrace{\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}}_{\text{CHANGE OF ORIGIN}},$$

WHERE A IS THE MATRIX

$$A = \begin{pmatrix} \sqrt{\frac{\sigma_X^2(1+\rho)}{2}} & -\sqrt{\frac{\sigma_X^2(1-\rho)}{2}} \\ \sqrt{\frac{\sigma_Y^2(1+\rho)}{2}} & \sqrt{\frac{\sigma_Y^2(1-\rho)}{2}} \end{pmatrix}$$

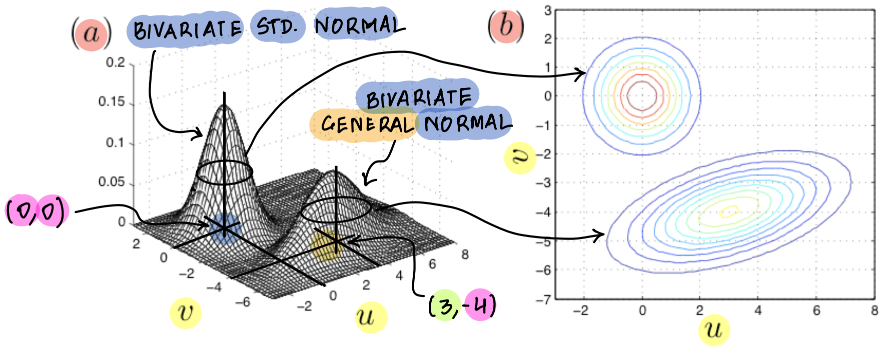

Figure 4.27: (a) Mesh plots of both the standard bivariate normal, and the bivariate normal with $\mu_X = 3, \mu_Y = -4, \sigma_X = 2, \sigma_Y = 1, \rho = 0.5$, shown on the same axes. (b) Contour plots of the same pdfs.

KEY PROPERTIES OF THE BIVARIATE NORMAL DISTRIBUTION

GENERAL CASE

PROPOSITION. SUPPOSE X AND Y HAVE THE BIVARIATE NORMAL PDF WITH PARAMETERS μ_X, μ_Y, σ_X, σ_Y, AND ρ. THEN

DEFINITION ON PAGE 1

(a) $X \sim N(\mu_X, \sigma_X^2)$ AND $Y \sim N(\mu_Y, \sigma_Y^2)$

(b) $\underbrace{aX + bY \sim \text{NORMAL DISTRIBUTION} \quad \forall\, a, b \in \mathbb{R}}$

CORRELATION COEFFICIENT BETWEEN X AND Y

(c) $\rho_{X,Y} = \rho$

MIN MSE LINEAR ESTIMATOR

(d) X AND Y ARE INDEPENDENT IF AND ONLY IF $\rho = 0$.

MIN MSE UNCONSTRAINED ESTIMATOR

(e) FOR ESTIMATION OF Y FROM X, $L^*(X) = g^*(X)$.

(f) $Y \mid X = u \sim N(\rho u, \sigma_e^2)$, WHERE σ_e^2 = MSE OF $L^*(X)$

PROOF: FIRST NOTE THAT IT IS SUFFICIENT TO PROVE THE ABOVE PROPOSITION FOR THE CASE $\mu_X = \mu_Y = 0$ AND $\sigma_X^2 = \sigma_Y^2 = 1$ AS THE GENERAL CASE IS SIMPLY A LINEAR TRANSFORMATION (CHANGE OF ORIGIN AND SCALE) OF THIS SPECIAL CASE. FURTHERMORE THE CORRELATION COEFFICIENT IS ALSO NOT AFFECTED BY THE CHANGE OF ORIGIN AND SCALE.

(a) $X \sim N(\mu_X, \sigma_X^2)$ AND $Y \sim N(\mu_Y, \sigma_Y^2)$

IN THIS CASE:

$$\underbrace{f_{X,Y}(u,v)}_{\text{JOINT PDF OF } X \text{ AND } Y} = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\overbrace{u^2+v^2-2\rho uv}^{\overbrace{u^2}^{u^2(1-\rho^2)} + \underbrace{v^2-2\rho uv+\rho^2u^2}_{(v-\rho u)^2} - \rho^2u^2}}{2(1-\rho^2)}\right) ; \quad |\rho| < 1$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)}_{\text{PDF OF } N(0,1) \,:\, \underbrace{f_X(u)}_{\text{MARGINAL PDF OF } X}} \underbrace{\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right)}_{\text{PDF OF } N(\overbrace{\rho u}^{E(Y|X=u)},\,(1-\rho^2)) \,:\, \underbrace{f_{Y|X}(u|v)}_{\text{CONDITIONAL PDF OF } Y|X=u}} ; \quad |\rho| < 1$$

$\Rightarrow$ $E(X) = \mu_X$ AND $Var(X) = \sigma_X^2$

AND SIMILARLY, $E(Y) = \mu_Y$ AND $Var(Y) = \sigma_Y^2$

THIS PROVES (a).

(b) $aX + bY \sim$ NORMAL DISTRIBUTION $\forall\ a, b \in \mathbb{R}$

FROM STANDARD 2-D NORMAL TO GENERAL 2-D NORMAL DISCUSSION, WE KNOW THAT THE CLASS OF BIVARIATE NORMAL PDFS IS PRESERVED UNDER LINEAR TRANSFORMATION CORRESPONDING TO MULTIPLICATION OF $\begin{pmatrix} X \\ Y \end{pmatrix}$ BY A MATRIX A IF $\det A \neq 0$. GIVEN a, b, CHOOSE c, d SUCH THAT

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \overbrace{\det A}^{ad-bc} \neq 0$$

$$\Rightarrow A \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix} \sim \text{BIVARIATE GAUSSIAN}$$

USING PART (a): $aX + bY \sim$ UNIVARIATE GAUSSIAN. THIS PROVES (b).

(c) $\rho_{X,Y} = \rho$

$$\rho_{X,Y} = \frac{\text{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - \overset{0}{\mu_X}\,\overset{0}{\mu_Y}}{\underset{1}{\sigma_X}\,\underset{1}{\sigma_Y}}$$

$$\Rightarrow \rho_{X,Y} = E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\, \underbrace{f_{XY}(u,v)}_{\text{PART (a)}}\, dv\, du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\, \overbrace{f_X(u)\, f_{Y|X}(v|u)}\, dv\, du$$

$$= \int_{-\infty}^{\infty} u f_X(u) \underbrace{\int_{-\infty}^{\infty} v f_{Y|X}(v|u)\,dv}_{\rho u \text{ USING PART (a) AS } Y|X \sim N(\rho u,\, (1-\rho^2))} du$$

$$= \int_{-\infty}^{\infty} u f_X(u)\, \rho u \, du$$

$$= \rho \underbrace{\int_{-\infty}^{\infty} u^2 f_X(u)\, du}_{Var(X) = 1 \text{ USING PART (a) AS } X \sim N(0,1)}$$

$$= \rho$$

THIS PROVES (c).

(d) X AND Y ARE INDEPENDENT IF AND ONLY IF $\rho = 0$.

→ IF X AND Y ARE INDEPENDENT THEN $\rho = 0$:

ALWAYS TRUE AS INDEPENDENCE IMPLIES UNCORRELATEDNESS

→ IF $\rho = 0$ THEN X AND Y ARE INDEPENDENT :

CONSIDER $f_{XY}(u,v) = \underbrace{f_X(u)}\,\underbrace{f_{Y|X}(v|u)}$: ALWAYS TRUE

$$\text{PART (a)} \rightarrow N(0,1) \uparrow f_X(u), \quad \text{PART (a)} \rightarrow \underbrace{N(\rho u,\, 1-\rho^2)}_{\text{WHEN } \rho_{X,Y} = \rho = 0} \rightarrow N(0,1), \quad N(\rho u,\, 1-\rho^2) \uparrow f_{Y|X}(v|u)$$

THIS PROVES (d).

(e) FOR ESTIMATION OF Y FROM X, $L^*(X) = g^*(X)$.

LECTURE 37

RECALL THAT MIMIMUM MSE LINEAR ESTIMATOR IS

$$L^*(u) = \underset{0}{\mu_Y} + \underset{\rho}{\rho_{X,Y}}\, \underset{1}{\sigma_Y} \left(\frac{u - \overset{0}{\mu_X}}{\underset{1}{\sigma_X}} \right)$$

$$L^*(u) = \rho u \qquad \cdots (1)$$

LECTURE 37

RECALL THAT MIMIMUM MSE UNCONSTRAINED ESTIMATOR IS

$$g^*(u) = E(Y \mid X = u)$$

$$= \rho u \quad \text{AS} \quad Y \mid X = u \sim N(\rho u,\ 1 - \rho^2) \quad \text{USING PART (a)} \qquad \cdots (2)$$

(1) AND (2) $\Rightarrow$ $L^*(u) = g^*(u) = \rho u$. THIS PROVES (e).

(f) $Y \mid X = u \sim N(\rho u, \sigma_e^2)$, WHERE σ_e^2 = MSE OF $L^*(X)$

USING PART (a), WE HAVE $Y \mid X = u \sim N(\rho u,\ 1 - \rho^2)$

TO PROVE (f), WE NEED TO SHOW THAT MSE OF $L^*(X) = 1 - \rho^2$

LECTURE 37

RECALL THAT MSE OF $L^*(X) = \overset{1}{\sigma_Y^2}\left(1 - \overset{\rho^2}{\rho_{X,Y}^2}\right)$

$$= 1 - \rho^2$$

THIS PROVES (f).