ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 35 : FUNCTIONS OF JOINTLY DISTRIBUTED RANDOM VARIABLES

- TOPICS TO COVER (BASED ON CH 4.6 - 4.7)

  → INTRODUCTION

  → MAX OF JOINTLY DISTRIBUTED RANDOM VARIABLES

  → LINEAR FUNCTIONS OF JOINTLY DISTRIBUTED RANDOM VARIABLES

→ INTRODUCTION

WE ARE INTERESTED IN UNDERSTANDING THE DISTRIBUTION OF FUNCTIONS OF JOINTLY DISTRIBUTED RANDOM VARIABLES.

RANDOM VECTOR (OF SIZE 2)

(X, Y) ~ JOINTLY DISTRIBUTED RV

DEFINE $g(X, Y)$: FUNCTION OF (X, Y).

A CHOICE OF $g(\cdot, \cdot)$

WE HAVE ALREADY SEEN THE $\underbrace{\text{DISTRIBUTION OF } X+Y}_{\text{CONVOLUTION}}$, I.E., $g(u, v) = u + v$.

→ MAX OF JOINTLY DISTRIBUTED RANDOM VARIABLES

DEFINE $W = \max(X, Y)$ X, Y ARE INDEPENDENT CONT.-TYPE RVS

HOW TO EXPRESS f_W IN TERMS OF f_X AND f_Y ?

CONSIDER $F_W(t) = P\{W \le t\}$

$$= P\{\underbrace{\max(X,Y) \le t}\}$$

$$\Updownarrow$$

$$= P\{\overbrace{X \le t,\ Y \le t}\}$$

INDEPENDENCE

$$= P\{X \le t\}\,P\{Y \le t\}$$

$$= F_X(t) \cdot F_Y(t)$$

DIFFERENTIATING WITH RESPECT TO t: $\frac{d}{dx} h_1(x) h_2(x) = h_2(x) \frac{d}{dx} h_1(x) + h_1(x) \frac{d}{dx} h_2(x)$

$$f_W(t) = F_Y(t) \cdot f_X(t) + F_X(t)\, f_Y(t)$$

EXAMPLE: X, Y EACH FOLLOW EXP(1) INDEPENDENTLY. FIND THE PDF OF $W = \max(X, Y)$.

SOLUTION: $F_X(u) = \begin{cases} 1 - e^{-u}, & u \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases} \Rightarrow f_X(u) = \begin{cases} e^{-u}, & u \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$

$$F_Y(v) = \begin{cases} 1 - e^{-v}, & v \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases} \Rightarrow f_Y(v) = \begin{cases} e^{-v}, & v \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$$\therefore \quad f_W(t) = \begin{cases} (1 - e^{-t})\, e^{-t} + (1 - e^{-t})\, e^{-t}, & u \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

QUESTION: VERIFY THAT f_W IS INDEED A VALID DENSITY! $\rightsquigarrow$ $f_W(t) = \begin{cases} 2(1 - e^{-t})\, e^{-t}, & u \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$

→ LINEAR FUNCTIONS OF JOINTLY DISTRIBUTED RANDOM VARIABLES

WE CONSIDER A FUNCTION OF (X,Y) AS $\overbrace{(W, Z)}^{\text{A RANDOM VECTOR}} = g(\underbrace{X,Y}_{\text{A RANDOM VECTOR}})$.

THE VALUE OF THE FUNCTION INVOLVES TWO RVS

LET W AND Z BOTH ARE LINEAR FUNCTIONS OF X AND Y, I.E.,

WE WANT $f_{W,Z}$ IN TERMS OF $f_{X,Y}$ (GIVEN!)

$$\begin{cases} W = aX + bY \\ Z = cX + dY \end{cases}$$

WHERE a, b, c, d ARE CONSTANTS.

EQUIVALENTLY:

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{MATRIX OF LINEAR TRASFORMATION}} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

WE CAN WRITE THE UNDERLYING FUNCTION AS

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$$

THE DETERMINANT OF A, det(A) IS DEFINED AS $ad - bc$. IF $\det(A) \neq 0 \Rightarrow$ A^{-1} EXISTS. HENCE,

$$\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

WHERE $A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

PROPOSITION: LET $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, WHERE $\begin{pmatrix} X \\ Y \end{pmatrix}$ HAS PDF $f_{X,Y}$, and A IS A MATRIX WITH $\det(A) \neq 0$. THEN $\begin{pmatrix} W \\ Z \end{pmatrix}$ HAS JOINT PDF GIVEN BY

$$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

EXAMPLE: $W = X - Y$ AND $Z = X + Y$

$$\Rightarrow \begin{pmatrix} W \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\det(A) = 1 - (-1) = 2 \neq 0 \Rightarrow A^{-1} \text{ EXISTS}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$\therefore f_{W,Z}(\alpha,\beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

$$\Rightarrow f_{W,Z}(\alpha,\beta) = \frac{1}{2} f_{X,Y}\left(\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$$

$$f_{W,Z}(\alpha,\beta) = \frac{1}{2} f_{X,Y}\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right) \qquad (\alpha,\beta) \in \mathbb{R}^2$$

EXERCISE: OBTAIN THE MARGINAL OF $Z = X+Y$ IN THE ABOVE EXAMPLE AND SHOW THAT IT IS SAME AS THE CONVOLUTION FORMULA FROM LECTURE 33.

EXAMPLE : $W = X + Y$ AND $Z = Y$

$$\Rightarrow \begin{pmatrix} W \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\det(A) = 1 - (0) = 1 \neq 0 \Rightarrow A^{-1} \text{ EXISTS}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)$$

$$\Rightarrow f_{W,Z}(\alpha, \beta) = f_{X,Y}\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)$$

$$f_{W,Z}(\alpha, \beta) = f_{X,Y}(\alpha - \beta, \beta) \qquad (\alpha, \beta) \in \mathbb{R}^2$$

IF X AND Y ARE INDEPENDENT :

$$\Rightarrow f_{W,Z}(\alpha, \beta) = f_X(\alpha - \beta) f_Y(\beta) \qquad (\alpha, \beta) \in \mathbb{R}^2$$

$$\therefore \quad f_W(\alpha) = \int_{-\infty}^{\infty} f_{W,Z}(\alpha, \beta) \, d\beta$$

$$= \underbrace{\int_{-\infty}^{\infty} f_X(\alpha - \beta) f_Y(\beta) \, d\beta}_{f_X * f_Y}$$

$f_X * f_Y$: CONVOLUTION