ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 33 : DISTRIBUTION OF SUMS OF RANDOM VARIABLES

- TOPICS TO COVER (BASED ON CH 4.3)
  - → DISTRIBUTION OF SUMS OF RANDOM VARIABLES

→ DISTRIBUTION OF SUMS OF RANDOM VARIABLES

X AND Y ARE TWO RVS DEFINED OVER THE SAME PROB. SPACE $(\Omega, \mathcal{F}, P)$.

$$X : \Omega \longrightarrow \mathbb{R} \quad \text{AND} \quad Y : \Omega \longrightarrow \mathbb{R}$$

WE ARE INTERESTED IN UNDERSTANDING THE DISTRIBUTION OF THEIR SUM, $S := X+Y$.

→ DISTRIBUTION OF SUMS OF DISCRETE-TYPE RANDOM VARIABLES

CONSIDER $S = X+Y$ WHERE X AND Y ARE DISCRETE-TYPE.

ASSUMPTION: WITHOUT LOSS OF GENERALITY X AND Y ARE INTEGER-TYPE.

$$p_S(k) = P\{X+Y = k\}$$

$$= \sum_j P\{X = j, Y = k-j\}$$

LAW OF TOTAL PROBABILITY

$\{X = j, Y = k-j\}$ j INTEGER ARE DISJOINT AND EXHAUSTIVE EVENTS

$$= \sum_j p_{X,Y}(j, k-j) \quad \ldots (1)$$

IF X AND Y ARE INDEPENDENT : $p_{X,Y}(j, k-j) = p_X(j)\, p_Y(k-j)$.

HENCE,

$$p_S(k) = \sum_j p_X(j)\, p_Y(k-j) \quad \ldots (2)$$

EQUATION (2) DEFINES THE CONVOLUTION OPERATION $(*)$, I.E.,

$$p_S = p_X * p_Y \quad \text{IF } S = X+Y \text{ AND } X, Y \text{ ARE INDEP.}$$

EXAMPLE: $X \sim$ BINOMIAL (m, p) AND $Y \sim$ BINOMIAL (n, p)

X AND Y ARE INDEPENDENT

BY THE CONVOLUTION FORMULA:

$$p_S(k) = \sum_j p_X(j)\, p_Y(k-j)$$

$$= \sum_j \binom{m}{j} p^j (1-p)^{m-j} \binom{n}{k-j} p^{k-j} (1-p)^{n-k+j}$$

$$= p^k (1-p)^{m+n-k} \underbrace{\left(\sum_j \binom{n}{j} \binom{m}{k-j} \right)}$$

WHY? EXERCISE!

$$= \overbrace{\binom{m+n}{k}} p^k (1-p)^{m+n-k}$$

$$\Rightarrow \quad S \sim \text{BINOMIAL}(m+n, p)$$

CONVENTIONS: $\binom{m}{j} = 0$ IF $j > m$ AND $\binom{n}{k-j} = 0$ IF $k-j > n$

→ DISTRIBUTION OF SUMS OF CONT. - TYPE RANDOM VARIABLES

CONSIDER $S = X + Y$ WHERE X AND Y ARE CONT. - TYPE

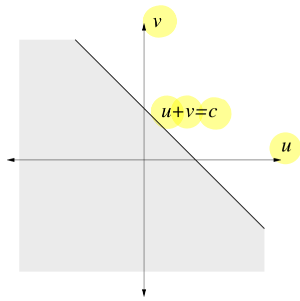

Figure 4.14: Shaded region for computation of $F_S(c)$, where $S = X + Y$.

CONSIDER :

$$F_S(c) = P\{S \le c\} = P\{X+Y \le c\} = \bigcup_u P\{X \le u, Y \le c-u\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\, dv\, du$$

$$\therefore \quad f_S(c) = \frac{d}{dc} F_S(c)$$

$$= \frac{d}{dc} \int_{-\infty}^{\infty} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\, dv\, du$$

FUBINI'S RULE (BEYOND OUR SCOPE)

$$= \int_{-\infty}^{\infty} \underbrace{\frac{d}{dc} \int_{-\infty}^{c-u} f_{X,Y}(u,v)\, dv}\, du$$

LEIBNIZ'S RULE (BEYOND OUR SCOPE)

$$f_S(c) = \int_{-\infty}^{\infty} f_{X,Y}(u, c-u)\, du \quad \cdots (3)$$

IF X AND Y ARE INDEPENDENT : $f_{X,Y}(u, c-u) = f_X(u)\, f_Y(c-u)$

HENCE,

$$f_S(c) = \int_{-\infty}^{\infty} f_X(u)\, f_Y(c-u)\, du \qquad \cdots (4)$$

EQUATION (4) DEFINES THE CONVOLUTION OPERATION ($*$), I.E.,

$f_S = f_X * f_Y$ IF $S = X + Y$ AND X, Y ARE INDEP.

EXAMPLE: $X \sim \text{NORMAL}(0, \sigma_1^2)$ AND $Y \sim \text{NORMAL}(0, \sigma_2^2)$

X AND Y ARE INDEPENDENT.

BY THE CONVOLUTION FORMULA:

LOOK UP:

→ RADIAL BASIS KERNELS

→ SUPPORT VECTOR MACHINES

$$f_S(c) = \int_{-\infty}^{\infty} f_X(u)\, f_Y(c-u)\, du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma_1^2}} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(c-u)^2}{2\sigma_2^2}} du$$

$$= \frac{1}{\sigma_1\sigma_2 2\pi} \int_{-\infty}^{\infty} e^{-\left(\frac{u^2}{2\sigma_1^2} + \frac{(c-u)^2}{2\sigma_2^2}\right)} du \qquad \cdots (5)$$

CONSIDER:

$$-\left(\frac{u^2}{2\sigma_1^2} + \frac{(c-u)^2}{2\sigma_2^2}\right) = -\left(\frac{u^2}{2\sigma_1^2} + \frac{c^2 + u^2 - 2cu}{2\sigma_2^2}\right)$$

$$= -\left(u^2\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right) - \frac{cu}{\sigma_2^2} + \frac{c^2}{2\sigma_2^2}\right)$$

$$= -\left(\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}u^2 - \frac{c}{\sigma_2^2}u + \frac{c^2}{2\sigma_2^2}\right)$$

$$= -\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(u^2 - \frac{2c\sigma_1^2}{\sigma_1^2+\sigma_2^2}u\right) - \frac{c^2}{2\sigma_2^2}$$

$$= -\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\left(u^2 - \frac{c\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 - \left(\frac{c\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2\right) - \frac{c^2}{2\sigma_2^2}$$

$$= -\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(u^2 - \frac{c\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 + \underbrace{\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\frac{c\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 - \frac{c^2}{2\sigma_2^2}}_{= \frac{-c^2}{2(\sigma_1^2+\sigma_2^2)}}$$

$$\therefore \quad f_S(c) = \frac{1}{\sigma_1\sigma_2 2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(u^2-\frac{c\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 - \frac{c^2}{2(\sigma_1^2+\sigma_2^2)}}\,du$$

$$= \frac{1}{\sigma_1\sigma_2 2\pi} e^{-\frac{c^2}{2(\sigma_1^2+\sigma_2^2)}}\int_{-\infty}^{\infty} e^{-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(u^2-\frac{c\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}\,du \quad \cdots (6)$$

WE KNOW THAT :

$$(7): \int_{-\infty}^{\infty} e^{-b(x-a)^2} = \sqrt{\frac{\pi}{b}}$$

(WHY ?)

IF $X \sim N(\mu, \sigma^2)$:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sigma\sqrt{2\pi}$$

SET $b = \frac{1}{2\sigma^2}$ IN ABOVE TO GET (7)

IN (6) : $b = \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}$

$$\Rightarrow f_S(c) = \frac{1}{\sigma_1\sigma_2 2\pi} e^{-\frac{c^2}{2(\sigma_1^2+\sigma_2^2)}}\sqrt{\frac{\pi 2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{c^2}{2(\sigma_1^2+\sigma_2^2)}} \quad : \text{NORMAL}(0, (\sigma_1^2+\sigma_2^2))$$