ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 32 : JOINT PROBABILITY DENSITY FUNCTIONS

- TOPICS TO COVER (BASED ON CH 4.3)
  - → JOINT PDFS

→ JOINT PDFS

X AND Y ARE TWO CONT.-TYPE RVS DEFINED OVER THE SAME PROB. SPACE $(\Omega, \mathcal{F}, P)$. $X : \Omega \longrightarrow \mathbb{R}$ AND $Y : \Omega \longrightarrow \mathbb{R}$

THE JOINT PDF $f_{X,Y}$ OF X AND Y IS DEFINED AS

$$\underbrace{F_{X,Y}(u_0, v_0)}_{\text{'JOINT' CDF OF } (X,Y)} = \underbrace{\int_{-\infty}^{u_0} \int_{-\infty}^{v_0}}_{\substack{\text{INTEGRATION OVER THE SHADED REGION IN FIG. 4.1} \\ \text{IN LECTURE 30}}} f_{X,Y}(u,v)\, dv\, du \qquad \text{FOR ANY } (u_0, v_0) \in \mathbb{R}^2$$

LET $R = (a, b] \times (c, d]$; $a < b$, $c < d$ AS SHOWN IN FIG. 4.2 (LECTURE 30)

$$P\{(X,Y) \in R\} = \iint_R f_{X,Y}(u,v)\, dv\, du \qquad \ldots (*)$$

$(*)$ HOLDS FOR ANY R THAT HAS A PIECEWISE DIFFERENTIABLE BOUNDARY.

NOTE: IF $R = \mathbb{R}^2$: $P\{(X,Y) \in \mathbb{R}^2\} = \iint_{\mathbb{R}^2} f_{X,Y}(u,v)\, dv\, du = 1$

FOR ANY FUNCTION g , THE EXPECTATION OF THE RV $g(X,Y)$ IS GIVEN BY

$$E[g(X,Y)] = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}}_{\mathbb{R}^2} g(u,v)\, f_{X,Y}(u,v)\, dv\, du \qquad \text{(LOTUS)}$$

THE ABOVE IMPLIES LINEARITY OF EXPECTATION :

$$g(X,Y) = aX + bY + c$$

$$E[aX + bY + c] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [au + bv + c]\, f_{X,Y}(u,v)\, dv\, du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [au + bv + c]\, f_{X,Y}(u,v)\, dv\, du$$

$$= \underbrace{a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\, \overbrace{f_{X,Y}(u,v)}^{\text{JOINT PDF OF X AND Y}} dv\, du}_{EX \text{ ! WHY? SET } g(X,Y) = X \text{ IN LOTUS}} + \underbrace{b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v\, f_{X,Y}(u,v)\, dv\, du}_{EY}$$

$$+ \underbrace{c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv\, du}_{\lim\limits_{u\to\infty}\lim\limits_{v\to\infty} F_{X,Y}(u,v) = 1}$$

$$EX = \int_{-\infty}^{\infty} u\, \underbrace{f_X(u)}_{\text{MARGINAL PDF OF X}}\, du$$

$$= a\,EX + b\,EY + c$$

NECESSARY AND SUFFICIENT CONDITIONS FOR f TO BE A VALID JOINT PDF :

PDF.1 FOR ANY $(u,v) \in \mathbb{R}^2$, $f(u,v) \geq 0$.

PDF.2 THE INTEGRAL OF f OVER $\mathbb{R}^2$ IS ONE, I.E. $\iint_{\mathbb{R}^2} f_{X,Y}(u,v)\, dv\, du = 1$.

SUPPORT OF $f_{X,Y}$: $\{ (u,v) \in \mathbb{R}^2 : f_{X,Y}(u,v) > 0 \}$

MARGINAL PDFS OF $f_{X,Y}$:

THE PDFS OF X AND Y ALONE ARE CALLED THE MARGINALS PDFS OF A JOINT PDF $f_{X,Y}$. WE CAN OBTAIN THE MARGINAL PDFS AS FOLLOWS:

CONSIDER $F_X(u_0) := P\{X \le u_0\}$

$$\lim_{v\to\infty} F_{X,Y}(u_0, v) = P\{X \le u_0, \overbrace{Y < \infty}^{\text{ALWAYS TRUE : } \Omega}\}$$

$$= \int_{-\infty}^{u_0}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv\, du \quad \cdots (1)$$

RIEMANN INTEGRAL (SEARCH FOR LEBESGUE INTEGRAL)

ALSO RECALL $F_X(u_0) := \int_{-\infty}^{u_0} f_X(u_0)\, du \quad \cdots (2)$

COMPARING (1) AND (2) : $f_X(u_0) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv \quad \cdots (3)$

SIMILARLY, $f_Y(v_0) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, du$

EXPECTATIONS OF X AND Y :

$$EX = \int_{-\infty}^{\infty} u\, f_X(u)\, du$$

$$(3) \Rightarrow \quad EX = \int_{-\infty}^{\infty} u \left(\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv \right) du \quad \cdots (4)$$

ALSO $X = g(X,Y)$

$\therefore$ LOTUS : $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\, f_{X,Y}(u,v)\, dv\, du$

$$\therefore \quad EX = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\, f_{X,Y}(u,v)\, dv\, du \qquad \cdots (5)$$

(4) AND (5) YIELD THE SAME RESULT.

SIMILARY: $EY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v\, f_{X,Y}(u,v)\, dv\, du$

CONDITIONAL PDFS OF X AND Y :

$$f_{X|Y}(u \mid v_0) := \begin{cases} \dfrac{f_{X,Y}(u, v_0)}{f_Y(v_0)}, & \text{IF } f_Y(v_0) > 0, \\ \text{UNDEFINED}, & \text{OTHERWISE}. \end{cases} \qquad u \in \mathbb{R}$$

$$f_{Y|X}(v \mid u_0) := \begin{cases} \dfrac{f_{X,Y}(u_0, v)}{f_X(u_0)}, & \text{IF } f_X(u_0) > 0, \\ \text{UNDEFINED}, & \text{OTHERWISE}. \end{cases} \qquad u \in \mathbb{R}$$

CONDITIONAL EXPECTATIONS :

$E(X \mid Y = v) = \int_{-\infty}^{\infty} u\, f_{X|Y}(u|v)\, du$ AND $E(Y \mid X = u) = \int_{-\infty}^{\infty} v\, f_{Y|X}(v|u)\, dv$

NOTE: $E(X|Y=v)$ IS A DETERMINISTIC FUNCTION OF v.

DENOTE $g(v) := E(X|Y=v)$ FOR ALL $v \in \mathbb{R}$

APPLY g TO Y: $g(Y) = E(X|Y)$

A FUNCTION OF Y

AS Y IS A RANDOM VARIABLE!

$\Rightarrow$ $g(Y)$ IS A RANDOM VARIABLE!

SIMILARLY, DEFINE $h(u) = E(Y|X=u)$

AS X IS A RANDOM VARIABLE!

$\Rightarrow$ $h(X) = E(X|Y)$ IS A RADOM VARIABLE!

EXAMPLE: SUPPOSE X AND Y THE JOINT PDF:

$$f_{X,Y}(u,v) = \begin{cases} c(1-u-v), & \text{IF } u \geq 0,\ v \geq 0,\ u+v \leq 1, \\ 0, & \text{OTHERWISE.} \end{cases}$$

WHERE c IS A CONSTANT TO BE DETERMINED.

- DARW THE JOINT PDF AND ITS SUPPORT.
- OBTAIN THE CONSTANT c.
- OBTAIN THE MARGINAL PDF OF X.
- OBTAIN THE CONDITIONAL PDF OF Y GIVEN X.

- DARW THE JOINT PDF AND ITS SUPPORT.

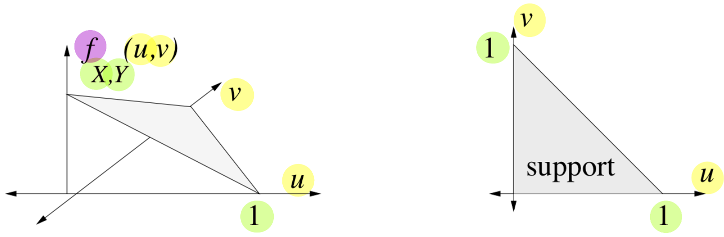

Figure 4.4: The pdf (4.11) and its support.

- OBTAIN THE CONSTANT c.

CONSIDER $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du = 1$$

$$\Rightarrow \int_{u=0}^{\infty}\int_{v=0}^{\infty} c(1-u-v)\,dv\,du = 1$$

$(u+v \leq 1)$

$u+v \leq 1 \Rightarrow u \leq 1$ AND $v \leq 1-u$

$$\Rightarrow \int_{u=0}^{1}\int_{v=0}^{1-u} c(1-u-v)\,dv\,du = 1$$

$$\Rightarrow c\int_{u=0}^{1}\int_{v=0}^{1-u} dv\,du - c\int_{u=0}^{1}\int_{v=0}^{1-u} u\,dv\,du - c\int_{u=0}^{1}\int_{v=0}^{1-u} v\,dv\,du = 1$$

$$\Rightarrow c\int_{0}^{1} v\Big|_{0}^{1-u} du - c\int_{0}^{1} u\,v\Big|_{0}^{1-u} du - c\int_{0}^{1} \frac{v^2}{2}\Big|_{0}^{1-u} du = 1$$

$$\Rightarrow c\int_{0}^{1} (1-u)\,du - c\int_{0}^{1} u(1-u)\,du - c\int_{0}^{1} \frac{(1-u)^2}{2}\,du = 1$$

$$\Rightarrow \quad c\left(u\Big|_0^1 - \frac{u^2}{2}\Big|_0^1\right) - c\left(\frac{u^2}{2}\Big|_0^1 - \frac{u^3}{3}\Big|_0^1\right) - \frac{c}{2}\left(u\Big|_0^1 - 2\frac{u^2}{2}\Big|_0^1 + \frac{u^3}{3}\Big|_0^1\right) = 1$$

$$\Rightarrow \quad c\left(1 - \frac{1}{2}\right) - c\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{c}{2}\left(1 - 1 + \frac{1}{3}\right) = 1$$

$$\Rightarrow \quad \frac{c}{2} - \frac{c}{2} + \frac{c}{3} - \frac{c}{6} = 1$$

$$\Rightarrow \quad \frac{c}{6} = 1 \quad \Rightarrow \quad c = 6$$

- OBTAIN THE MARGINAL PDF OF X.

$$f_X(u) = \int_{\substack{v:0 \\ u+v\le 1}}^{1} f_{X,Y}(u,v)\,dv$$

$$= 6\int_0^{1-u} (1-u-v)\,dv$$

$$= 6\left(\int_0^{1-u} dv - u\int_0^{1-u} dv - \int_0^{1-u} v\,dv\right)$$

$$= 6\left(v\Big|_0^{1-u} - u\,v\Big|_0^{1-u} - \frac{v^2}{2}\Big|_0^{1-u}\right)$$

$$= 6\left((1-u) - u(1-u) - \frac{(1-u)^2}{2}\right)$$

$$= 6\left((1-u)\left(1 - u - \frac{(1-u)}{2}\right)\right)$$

$$= 6\left((1-u) \frac{(1-u)}{2} \right)$$

$$f_X(u) = 3(1-u)^2 \qquad u \in (0,1)$$

- OBTAIN THE CONDITIONAL PDF OF Y GIVEN X.

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(v)}$$

$$= \begin{cases} \dfrac{6(1-u-v)}{3(1-u)^2}, & 0 \leq v < 1-u, \\ 0, & \text{OTHERWISE.} \end{cases}$$

INDEPENDENCE OF TWO CONT.-TYPE RVS :

TWO CONT.-TYPE RVS ARE INDEPENDENT IF

$$(1.1) \qquad f_{X,Y}(u,v) = f_X(u) \cdot f_Y(v) \qquad \forall \quad (u,v) \in \mathbb{R}^2$$

NOTE THAT WHEN (u,v) IS OUTSIDE SUPPORT OF $f_{X,Y}$, THE ABOVE CONDITION IS $0 = 0 \cdot 0$: TRIVIALLY TRUE.

HENCE IT IS SUFFICIENT TO VERIFY IT FOR THE SUPPORT OF $f_{X,Y}$:

$\{ (u_i, v_j) : i = 1, 2, \dots ; j = 1, 2, \dots \}$

FURTHER NOTE THAT : $f_{X,Y}(u,v) = f_X(u) \cdot f_{Y|X}(v|u)$ WHEN $f_X(u) > 0$

UNDER INDEPENDENCE : $f_{X,Y}(u,v) = f_X(u) \cdot f_Y(v) \quad \forall (u,v) \in \mathbb{R}^2$

$\therefore$ INDEPENDENCE IS EQUIVALENT TO

$$(1.2) \qquad f_{Y|X}(v|u) = f_Y(v) \qquad \text{WHEN } f_X(u) > 0$$

WHEN $f_X(u) = 0$, INDEPENDENCE HOLDS TRIVIALLY AS u IS OUTSIDE THE SUPPORT OF f_X AND ALSO (u,v) IS ALSO OUTSIDE THE SUPPORT OF $f_{X,Y}$ $\forall v$.

EXERCISE : FOR THE CONT.-TYPE RVS X AND Y IN PREVIOUS EXAMPLE, CHECK THEIR INDEPENDENCE USING CONDITIONS 1.1 AND 1.2 .

WORDS OF WISDOM

DUNNING - KRUGER EFFECT

WOW! : ONE KNOWS TOO LITTLE TO KNOW THAT THEY KNOW TOO LITTLE.