ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 31 : JOINT PROBABILITY MASS FUNCTIONS

- TOPICS TO COVER (BASED ON CH 4.2)

  → JOINT PMFS

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X AND Y ARE TWO DISCRETE-TYPE RVS DEFINED OVER THE SAME PROB. SPACE $(\Omega, \mathcal{F}, P)$. $X : \Omega \longrightarrow \mathbb{R}$ AND $Y : \Omega \longrightarrow \mathbb{R}$

THE JOINT PMF $p_{X,Y}$ OF X AND Y IS DEFINED AS

$$p_{X,Y}(u,v) := P\{X = u, Y = v\} \quad \text{FOR ANY } (u,v) \in \mathbb{R}^2$$

DETERMINES THE PROBS OF ALL EVENTS ASSOCIATED WITH X AND Y

AS X AND Y ARE BOTH DISCRETE-TYPE, THERE IS A FINITE OR COUNTABLE SET OF POSSIBLE VALUES OF X, DENOTE THEM BY $u_1, u_2, \ldots$

SIMILARLY, THERE IS A FINITE OR COUNTABLE SET OF POSSIBLE VALUES OF Y, DENOTE THEM BY $v_1, v_2, \ldots$

⇒ SUPPORT OF $p_{X,Y}$: $\{(u_i, v_j) : i = 1, 2, \ldots ; j = 1, 2, \ldots\}$ IS AT MOST A COUNTABLE SET!

CONSIDER THE EVENTS OF THE FORM $E_j = \{Y = v_j\}$ $j = 1, 2, \ldots$

NOTE THAT THESE EVENTS ARE MUTUALLY EXCLUSIVE.

ALSO CONSIDER THE EVENT $F = \{Y \notin \{v_1, v_2, \ldots\}\}$. NOTE THAT $P(F) = 0$.

E_j $j = 1, 2 \ldots$ AND F FORM A PARTITION OF Ω, I.E.,

- $E_j \cap E_k = \phi \quad \forall\, j \neq k$ AND $E_j \cap F = \phi \quad \forall\, j$
- $\left(\bigcup_j E_j\right) \cup F = \Omega$

$A_1, A_2, \ldots$: A PARTITION OF Ω THEN $B \subseteq \Omega$: $B = \bigcup_i BA_i \Rightarrow P(B) = \sum_i P(BA_i)$

HENCE USING THE $\overbrace{\text{LAW OF TOTAL PROBABILITY}}$:

$$\{X = u\} = \left(\bigcup_j \{X = u\} \cap E_j\right) \cup (\{X = u\} \cap F)$$

$$\Rightarrow P\{X = u\} = \sum_j P(\{X = u\} \cap E_j) + \underbrace{P(\{X = u\} \cap F)}_{=0 \text{ AS } P(F) = 0}$$

$$\Rightarrow P\{X = u\} = \sum_j P\{X = u, Y = v_j\}$$

PMF OF X : MARGINAL PMF OF $p_{X,Y}$ FOR X

I.E. $p_X(u) = \sum_j P\{X = u, Y = v_j\} = \sum_j p_{X,Y}(u, v_j)$

SIMILARLY, $p_Y(v) = \sum_i P\{X = u_i, Y = v\} = \sum_i p_{X,Y}(u_i, v)$

IN THIS CASE p_X AND p_Y ARE CALLED THE MARGINAL PMFS OF THE JOINT PMF p_{XY}

THE CONDITIONAL PMFS ARE ALSO DETERMINE BY THE JOINT PMF.

- CONDITIONAL PMF OF Y GIVEN $X = \underset{\text{FIXED}}{u_0}$: $p_{Y|X=u_0}$ IS

$$p_{Y|X=u_0}(v) = P\{Y = v \mid X = u_0\} \qquad P(A|B) = \frac{P(AB)}{P(B)} \text{ WHEN } P(B) > 0$$

ANOTHER NOTATION $\rightarrow$ $p_{Y|X}(v|u_0)$

$$= \frac{P\{Y = v,\ X = u_0\}}{P\{X = u_0\}} \quad \text{WHEN } P\{X = u_0\} > 0$$

I.E.,

$$p_{Y|X}(v|u_0) = \frac{p_{X,Y}(v, u_0)}{p_X(u_0)} \quad \text{WHEN } p_X(u_0) > 0$$

WHEN $p_X(u_0) = 0$, $p_{Y|X}(v|u_0)$ IS UNDEFINED.

- CONDITIONAL PMF OF X GIVEN $Y = v_0$: $p_{X|Y=v_0}$ IS

$$p_{X|Y}(u|v_0) = \frac{p_{X,Y}(u, v_0)}{p_Y(v_0)} \quad \text{WHEN } p_Y(v_0) > 0$$

WHEN $p_Y(v_0) = 0$, $p_{X|Y}(u|v_0)$ IS UNDEFINED.

EXAMPLE. LET (X,Y) HAVE THE JOINT PMF GIVEN BY TABLE 4.1

Table 4.1: A simple joint pmf. $p_{X,Y}$

	$X=1$	$X=2$	$X=3$	ROW SUM p_Y
$Y=3$	0.1	0.1		$0.2 \leftarrow P\{Y=3\}$
$Y=2$		0.2	0.2	$0.4 \leftarrow P\{Y=2\}$
$Y=1$		0.3	0.1	$0.4 \leftarrow P\{Y=1\}$
COLUMN SUM p_X	$0.1 \leftarrow P\{X=1\}$	$0.6 \leftarrow P\{X=2\}$	$0.3 \leftarrow P\{X=3\}$	1

1:
- SUM OF ROW SUMS
- SUM OF COLUMN SUMS
- SUM OF ALL ENTRIES IN THE TABLE

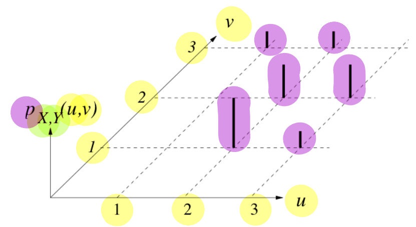

Figure 4.3: The graph of $p_{X,Y}$.

$$\rightarrow \quad p_X(u) = \sum_j P\{X=u, Y=v_j\} = \sum_j p_{X,Y}(u, v_j)$$

$$\Rightarrow \quad p_X(1) = 0.1, \quad p_X(2) = 0.3 + 0.2 + 0.1 = 0.6$$

$$p_X(3) = 0.1 + 0.2 = 0.3$$ "COLUMN SUMS"

$$\rightarrow \quad p_Y(v) = \sum_i P\{X=u_i, Y=v\} = \sum_i p_{X,Y}(u_i, v)$$

$$\Rightarrow \quad p_Y(1) = 0.3 + 0.1 = 0.4$$

$$p_Y(2) = 0.2 + 0.2 = 0.4$$ "ROW SUMS"

$$p_Y(3) = 0.1 + 0.1 = 0.2$$

$\rightarrow \quad P\{X=Y\} = p_{X,Y}(1,1) + p_{X,Y}(2,2) + p_{X,Y}(3,3)$

$= 0 + 0.2 + 0 = 0.2$

$\rightarrow \quad P\{X>Y\} = p_{X,Y}(2,1) + p_{X,Y}(3,1) + p_{X,Y}(3,2)$

$= 0.3 + 0.1 + 0.2 = 0.6$

$\rightarrow \quad p_{Y|X}(v|u_0) = \dfrac{p_{X,Y}(v,u_0)}{p_X(u_0)}$ WHEN $p_X(u_0) > 0$

$\therefore \quad p_{Y|X}(v|2) = \dfrac{p_{X,Y}(v,2)}{p_X(2)}$

$p_{Y|X}(1|2) = \dfrac{p_{X,Y}(1,2)}{p_X(2)} = \dfrac{0.3}{0.6} = \dfrac{1}{2}$

$p_{Y|X}(2|2) = \dfrac{p_{X,Y}(2,2)}{p_X(2)} = \dfrac{0.2}{0.6} = \dfrac{1}{3}$

$p_{Y|X}(3|2) = \dfrac{p_{X,Y}(3,2)}{p_X(2)} = \dfrac{0.1}{0.6} = \dfrac{1}{6}$

NECESSARY AND SUFFICIENT CONDITIONS FOR p TO BE A VALID JOINT PMF

- PMF.1 : p IS NON-NEGATIVE.
- PMF.2 : THERE ARE FINITE OR COUNTABLE SETS $\{u_1, u_2, \ldots\}$ AND $\{v_1, v_2, \ldots\}$ SUCH THAT $p(u,v) = 0$ IF $X \notin \{u_1, u_2, \ldots\}$ OR IF $Y \notin \{v_1, v_2, \ldots\}$
- PMF.3 : $\sum_i \sum_j p(u_i, v_j) = 1$

EXERCISE: FOR THE JOINT PMF IN THE PREVIOUS EXAMPLE, CHECK THE ABOVE PROPERTIES.

INDEPENDENCE OF TWO DISCRETE-TYPE RVS :

TWO DISCRETE-TYPE RVS ARE INDEPENDENT IF

$$p_{X,Y}(u,v) = p_X(u) \cdot p_Y(v) \quad \forall (u,v) \in \mathbb{R}^2 \tag{1.1}$$

NOTE THAT WHEN (u,v) IS OUTSIDE SUPPORT OF $p_{X,Y}$, THE ABOVE CONDITION IS $0 = 0 \cdot 0$: TRIVIALLY TRUE.

HENCE IT IS SUFFICIENT TO VERIFY IT FOR THE SUPPORT OF $p_{X,Y}$:

$\{(u_i, v_j) : i = 1, 2, \ldots ; j = 1, 2, \ldots\}$

FURTHER NOTE THAT: $p_{X,Y}(u,v) = p_X(u) \cdot p_{Y|X}(v|u)$ WHEN $p_X(u) > 0$

UNDER INDEPENDENCE: $p_{X,Y}(u,v) = p_X(u) \cdot p_Y(v) \quad \forall (u,v) \in \mathbb{R}^2$

$\therefore$ INDEPENDENCE IS EQUIVALENT TO

$$p_{Y|X}(v|u) = p_Y(v) \quad \text{WHEN } p_X(u) > 0 \tag{1.2}$$

WHEN $p_X(u) = 0$, INDEPENDENCE HOLDS TRIVIALLY AS u IS OUTSIDE THE SUPPORT OF p_X AND ALSO (u,v) IS ALSO OUTSIDE THE SUPPORT OF $p_{X,Y}$ $\forall v$.

EXERCISE: FOR THE DISCRETE-TYPE RVS X AND Y IN PREVIOUS EXAMPLE, CHECK THEIR INDEPENDENCE USING CONDITIONS 1.1 AND 1.2.