ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 30 : JOINTLY DISTRIBUTED RANDOM VARIABLES : JOINT CDFS

- TOPICS TO COVER (BASED ON CH 4.1)

  → JOINT CDFS

→ JOINT CDFS

X AND Y ARE TWO RVS DEFINED OVER THE SAME PROB. SPACE $(\Omega, \mathcal{F}, P)$

$X : \Omega \longrightarrow \mathbb{R}$ AND $Y : \Omega \longrightarrow \mathbb{R}$

THE JOINT CDF $F_{X,Y}$ OF X AND Y IS DEFINED AS

$F_{X,Y}(u_0, v_0) := P\{X \leq u_0 , Y \leq v_0\}$ FOR ANY $(u_0, v_0) \in \mathbb{R}^2$

= PROBABILITY OF $(X,Y)$ FALLING INTO THE SHADED REGION IN FIG. 4.1

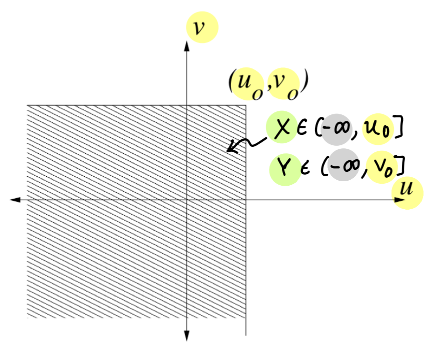


Figure 4.1: Region defining $F_{X,Y}(u_o, v_o)$.

NOTE THAT THE JOINT CDF DETERMINES THE PROBS OF ALL EVENTS CONCERNING X AND Y. FOR EXAMPLE IF R IS THE RECTANGULAR REGION $(a, b] \times (c, d]$, THEN

$$P\{(X,Y)\in R\} = F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c)$$

AS ILLUSTRATED IN FIG. 4.2.

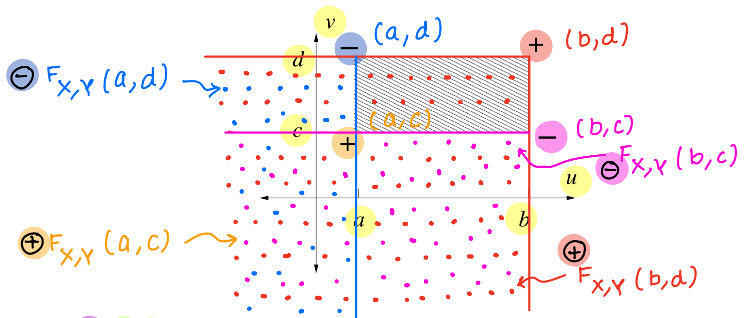


Figure 4.2: $P\{(X,Y)\in$ shaded region$\}$ is equal to $F_{X,Y}$ evaluated at the corners with signs shown.

THE JOINT CDF OF X AND Y ALSO DETERMINES THE PROBS OF EVENTS CONCERNING X ALONE, OR Y ALONE. HOW?

PROPOSITION. DEFINE $F_{X,Y}(u,\infty) := \lim_{v\to\infty} F_{X,Y}(u,v)$ FOR ANY FIXED $u$

$F_{X,Y}(\infty,v) := \lim_{u\to\infty} F_{X,Y}(u,v)$ " " " $v$

THEN $F_X(u) = F_{X,Y}(u,\infty)$ AND $F_Y(v) = F_{X,Y}(\infty,v)$

PROOF SKETCH: INTUITIVELY: $F_{X,Y}(u,\infty) := \lim_{v\to\infty} F_{X,Y}(u,v)$

$$= \lim_{v\to\infty} P\{X\le u, Y\le v\}$$

$$= P\{X\le u, \underbrace{Y\le\infty}_{\text{ALWAYS TRUE}: \Omega}\}$$

$$= P\{X\le u \cap \Omega\} \qquad \because A\Omega = A \ \forall A\subseteq\Omega$$

$$= P\{X\le u\} = F_X(u)$$

BY SYMMETRY, $F_Y(v) = F_{X,Y}(\infty, v)$.

## PROPERTIES OF JOINT CDFS:

F.1 $0 \leq F_{X,Y}(u,v) \leq 1 \quad \forall \quad (u,v) \in \mathbb{R}^2$

WHY? $F_{X,Y}$ IS A PROBABILITY $\in [0,1]$.

F.2 $F_{X,Y}(u,v)$ IS $\underbrace{\text{NONDECREASING IN } u}_{\text{FIX } v:\ u_1 \geq u_2:\ F_{X,Y}(u_1,v) \geq F_{X,Y}(u_2,v)}$ AND IS $\underbrace{\text{NONDECREASING IN } v}_{\text{FIX } u:\ v_1 \geq v_2:\ F_{X,Y}(u,v_1) \geq F_{X,Y}(u,v_2)}$

WHY? $F_{X,Y}$ IS A CUMULATIVE FUNCTION.

F.3 $F_{X,Y}(u,v)$ IS RIGHT-CONTINUOUS IN $u$ AND IS RIGHT-CONTINUOUS IN $v$

WHY? SIMILAR REASON AS SINGLE VAR DF.

F.4 IF $a < b$ AND $c < d$, $\underbrace{F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c) \geq 0}$

WHY? REFER TO FIG. 4.2. RECTANGLE INCLUSION-EXCLUSION INEQUALITY

F.5 $\lim_{u \to -\infty} F_{X,Y}(u,v) = 0 \quad \forall \ v \in \mathbb{R}$ AND $\lim_{v \to -\infty} F_{X,Y}(u,v) = 0 \quad \forall \ u \in \mathbb{R}$

WHY? EVENT ASSOCIATED WITH ONE VAR BECOMES $\phi$ AND $A\phi = \phi \quad \forall \ A \in \mathcal{F}$.

F.6 $\lim_{u \to \infty} \lim_{v \to \infty} F_{X,Y}(u,v) = 1$

WHY? EVENTS ASSOCIATED WITH BOTH RVS BECOME $\Omega$ AND $\Omega\Omega = \Omega$.

IT CAN BE SHOWN THAT ANY FUNCTION F SATISFYING THE ABOVE PROPERTIES IS THE JOINT CDF OF SOME PAIR OF RVS $(X,Y)$. F.1-F.6 CHARACTERIZE A JOINT CDF

EXAMPLE: CONSIDER THE FOLLOWING FUNCTION:

JOINT CDF OF TWO INDEP. EXP(1) RVS

$$F(u,v) = \begin{cases} (1-e^{-u})(1-e^{-v}) & , \quad u \geq 0 \;\; v \geq 0, \\ 0 & , \quad \text{OTHERWISE.} \end{cases}$$

LETS VERIFY THE ABOVE SIX PROPERTIES FOR F.

F.1 $0 \leq F(u,v) \leq 1 \quad \forall \quad (u,v) \in \mathbb{R}^2$

- $0 \leq F(u,v) \quad \forall \quad (u,v) \in \mathbb{R}^2$

$u, v \geq 0 \qquad F(u,v) = \underbrace{(1 - \overbrace{e^{-u}}^{\leq 1})}_{\geq 0}\underbrace{(1 - \overbrace{e^{-v}}^{\leq 1})}_{\geq 0}$

$\geq 0$

OTHERWISE $\quad F(u,v) = 0 \geq 0$

- $F(u,v) \leq 1 \quad \forall \quad (u,v) \in \mathbb{R}$

$u, v \geq 0 \qquad F(u,v) = (1-e^{-u})(1-e^{-v})$

AS $u, v \to \infty$ : $F(u,v) = \underbrace{(1 - \overbrace{e^{-u}}^{\to 0})}_{\to 1}\underbrace{(1 - \overbrace{e^{-v}}^{\to 0})}_{\to 1} \longrightarrow 1$

OTHERWISE $\quad F(u,v) = 0 \leq 1$

F.2 $F(u,v)$ IS NONDECREASING IN $u$ AND IS NONDECREASING IN $v$

DRAW THE FUNCTION : REFER TO THE DESMOS ILLUSTRATION.

ELSE TAKE PARTIAL DERIVATIVES WRT $u$ AND $v$ RESPECTIVELY AS FOLLOWS:

- $\dfrac{\partial F}{\partial u} = \dfrac{\partial}{\partial u}\left((1-e^{-u})(1-e^{-v})\right) = \underbrace{e^{-u}}_{>0}\underbrace{(1 - \overbrace{e^{-v}}^{<1})}_{>0} > 0 \qquad u > 0, \quad v > 0$

- $\dfrac{\partial F}{\partial v} = \dfrac{\partial}{\partial v}\left((1-e^{-u})(1-e^{-v})\right) = \underbrace{e^{-v}}_{>0}\underbrace{(1 - \overbrace{e^{-u}}^{<1})}_{>0} > 0 \qquad u > 0, \quad v > 0$

WHEN $F = 0 \qquad \dfrac{\partial F}{\partial u} = 0$ AND $\dfrac{\partial F}{\partial v} = 0$

**F.3** $F(u, v)$ IS RIGHT-CONTINUOUS IN $u$ AND IS RIGHT-CONTINUOUS IN $v$

$F$ IS CONTINUOUS IN $u$ AND IN $v$. HENCE, IT IS ALSO RIGHT-CONTINUOUS IN $u$ AND IN $v$.

**F.4** IF $a < b$ AND $c < d$, $F(b,d) - F(b,c) - F(a,d) + F(a,c) \geq 0$

CONSIDER $F(b,d) - F(b,c) - F(a,d) + F(a,c)$

$$= (1-e^{-b})(1-e^{-d}) - (1-e^{-b})(1-e^{-c})$$

$$- (1-e^{-a})(1-e^{-d}) + (1-e^{-a})(1-e^{-c})$$

$$= (1-e^{-b})\left[(1-e^{-d}) - (1-e^{-c})\right] + (1-e^{-a})\left[(1-e^{-c}) - (1-e^{-d})\right]$$

$$= (1-e^{-b})\left[e^{-c} - e^{-d}\right] - (1-e^{-a})\left[e^{-c} - e^{-d}\right]$$

$$= \left[e^{-c} - e^{-d}\right]\left[e^{-a} - e^{-b}\right]$$

SINCE $a < b$ : $e^{-a} > e^{-b} \Rightarrow e^{-a} - e^{-b} > 0$

$c < d$ : $e^{-c} > e^{-d} \Rightarrow e^{-c} - e^{-d} > 0$

$$\Rightarrow \left[e^{-c} - e^{-d}\right]\left[e^{-a} - e^{-b}\right] > 0$$

OTHERWISE WHEN $F = 0$ : THE INEQUALITY IS TRIVIALLY TRUE!

F.5 $\lim_{u \to -\infty} F(u,v) = 0 \quad \forall\ v \in \mathbb{R}$ AND $\lim_{v \to -\infty} F(u,v) = 0 \quad \forall\ u \in \mathbb{R}$

CONSIDER $\lim_{u \to -\infty} F(u,v) = \lim_{u \to -\infty} (1 - e^{-u})(1 - e^{-v})$ WHEN $u \geq 0 \quad v \geq 0$

$$= \lim_{u \to -\infty} (1 - e^{-u}) \lim_{u \to -\infty} (1 - e^{-v})$$

$$= (1 - \lim_{u \to -\infty} e^{-u})\ (1 - \lim_{u \to -\infty} e^{-v}) = (1 - 1)(1 - 1) = 0$$

OTHERWISE $F(u,v) = 0 \to 0$ AS $u \to -\infty \quad \forall\ u \in \mathbb{R}$

BY SYMMETRY, $\lim_{v \to -\infty} F(u,v) = 0 \quad \forall\ u \in \mathbb{R}$

F.6 $\lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = 1$

CONSIDER $\lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = \lim_{u \to \infty} \lim_{v \to \infty} (1 - e^{-u})(1 - e^{-v})$

$$= \lim_{u \to \infty} (1 - e^{-u}) \lim_{u \to \infty} (1 - e^{-v})$$

$$= (1 - \lim_{u \to \infty} e^{-u})\ (1 - \lim_{u \to \infty} e^{-v})$$

$$= (1 - 0)(1 - 0) = 1$$

OBTAINING $F_X$ AND $F_Y$ :

$$F_X(u) = \lim_{v \to \infty} F_{X,Y}(u,v) = \lim_{v \to \infty} (1 - e^{-u})(1 - e^{-v}) = (1 - e^{-u}) \lim_{v \to \infty} (1 - e^{-v})$$

$$= (1 - e^{-u})\ (1 - \underbrace{\lim_{v \to \infty} e^{-v}}_{=0}) = (1 - e^{-u}) \qquad u \geq 0$$

OTHERWISE $F_{X,Y}(u,v) = 0 \to 0$ AS $v \to \infty \quad \therefore \quad F_X(u) = 0$

BY SYMMETRY, $F_Y(v) = \begin{cases} 1 - e^{-v}, & v \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$

EXAMPLE: CONSIDER THE FOLLOWING FUNCTION:

CDF OF THE SUM OF TWO INDEP. EXP(1) RVS

$$F(u,v) = \begin{cases} 1 - e^{-(u+v)}, & u \geq 0 \quad v \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

LETS VERIFY THE ABOVE SIX PROPERTIES FOR F.

NOT A VALID 'JOINT' CDF THOUGH AS F.4 DOESN'T HOLD!

F.1 $0 \leq F(u,v) \leq 1 \quad \forall \quad (u,v) \in \mathbb{R}^2$

- $0 \leq F(u,v) \quad \forall \quad (u,v) \in \mathbb{R}^2$

$u, v \geq 0 \qquad F(u,v) = 1 - e^{-(u+v)}$

$\geq 0 \quad$ AS $\quad e^{-(u+v)} \leq 1 \quad$ FOR $u \geq 0$ AND $v \geq 0$

OTHERWISE $\quad F(u,v) = 0 \geq 0$

- $F(u,v) \leq 1 \quad \forall \quad (u,v) \in \mathbb{R}$

$u, v \geq 0 \qquad F(u,v) = 1 - e^{-(u+v)}$

AS $u, v \to \infty$ : $\quad e^{-(u+v)} \to 0 \quad \Rightarrow F(u,v) = 1 - e^{-(u+v)} \to 1$

OTHERWISE $\quad F(u,v) = 0 \leq 1$

F.2 $F(u,v)$ IS NONDECREASING IN $u$ AND IS NONDECREASING IN $v$

DRAW THE FUNCTION : REFER TO THE DESMOS ILLUSTRATION.

ELSE TAKE PARTIAL DERIVATIVES WRT $u$ AND $v$ RESPECTIVELY AS FOLLOWS:

- $\dfrac{\partial F}{\partial u} = \dfrac{\partial}{\partial u}\left(1 - e^{-(u+v)}\right) = e^{-(u+v)} > 0 \qquad u > 0, \quad v > 0$

- $\dfrac{\partial F}{\partial v} = \dfrac{\partial}{\partial v}\left(1 - e^{-(u+v)}\right) = e^{-(u+v)} > 0 \qquad u > 0, \quad v > 0$

WHEN $F = 0 \qquad \dfrac{\partial}{\partial u} F = 0$ AND $\dfrac{\partial}{\partial v} F = 0$

F.3 $F(u,v)$ IS RIGHT-CONTINUOUS IN $u$ AND IS RIGHT-CONTINUOUS IN $v$

$F$ IS CONTINUOUS IN $u$ AND IN $v$. HENCE, IT IS ALSO RIGHT-CONTINUOUS IN $u$ AND IN $v$.

F.4 IF $a < b$ AND $c < d$, $F(b,d) - F(b,c) - F(a,d) + F(a,c) \geq 0$

SHOW THAT THIS DOES NOT HOLD!

$\Rightarrow$ $F$ IS NOT A VALID JOINT CDF.

F.5 $\lim_{u \to -\infty} F(u,v) = 0 \quad \forall\, v \in \mathbb{R}$ AND $\lim_{v \to -\infty} F(u,v) = 0 \quad \forall\, u \in \mathbb{R}$

CONSIDER $\lim_{u \to -\infty} F(u,v) = \lim_{u \to -\infty} 1 - e^{-(u+v)}$ WHEN $u \geq 0$ $v \geq 0$

$$= 1 - \lim_{u \to -\infty} e^{-(u+v)}$$

$$= 1 - 1 = 0$$

OTHERWISE $F(u,v) = 0 \to 0$ AS $u \to -\infty \quad \forall\, u \in \mathbb{R}$

BY SYMMETRY, $\lim_{v \to -\infty} F(u,v) = 0 \quad \forall\, u \in \mathbb{R}$

F.6 $\lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = 1$

CONSIDER $\lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = \lim_{u \to \infty} \lim_{v \to \infty} (1 - e^{-(u+v)})$

$$= 1 - \lim_{u \to \infty} \lim_{v \to \infty} (e^{-(u+v)})$$

$$= 1$$