ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 3 : AXIOMS OF PROBABILITY

- TOPICS TO COVER (BASED ON CH 1.2)

→ AXIOMS OF PROBABILITY

→ AXIOMS OF PROBABILITY : $(\Omega, \mathcal{F}, P)$ — Ω: SAMPLE SPACE; $\mathcal{F}$: A SIGMA-FIELD / SIGMA-ALGEBRA OVER Ω

A CLASS OF SUBSETS OF Ω THAT IS CLOSED UNDER COMPLEMENTATION AND UNION

- RANDOM EXPERIMENT (RE) : ROLLING A DIE
- SAMPLE SPACE (Ω) : $\{1, 2, 3, 4, 5, 6\}$
- EVENT (E) : $E \subseteq \Omega$
- EVENT AXIOMS :

E.1. Ω IS AN EVENT, i.e., $\Omega \in \mathcal{F}$.

E.2. IF A IS AN EVENT, SO IS A^c, i.e., IF $A \in \mathcal{F}$, THEN $A^c \in \mathcal{F}$.

E.3. IF A and B ARE EVENTS, SO IS $A \cup B$, i.e., IF $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

E.G. Ω $\mathcal{F} = \{\phi, \Omega\}$ Q: IS $\mathcal{F}$ A VALID COLLECTION OF EVENTS?

E.1. $\Omega \in \mathcal{F}$: YES

E.2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

- $\phi \in \mathcal{F} \Rightarrow \phi^c = \Omega \in \mathcal{F}$
- $\Omega \in \mathcal{F} \Rightarrow \Omega^c = \phi \in \mathcal{F}$

} YES

E.3. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

- $\phi, \Omega \in \mathcal{F} \Rightarrow \phi \cup \Omega = \Omega \in \mathcal{F}$ YES

E.G. Ω $\mathcal{F}_2 = \{\phi, A, A^c, \Omega\}$ $A \subset \Omega$ Q: IS $\mathcal{F}_2$ A VALID COLLECTION OF EVENTS?

- PROPERTY E.4 $\phi \in \mathcal{F}$

PROOF: BY E.1, $\Omega \in \mathcal{F}$

BY E.2, $\Omega^c = \phi \in \mathcal{F}$

- PROPERTY E.5 $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

PROOF: BY DE-MORGAN'S LAW: $AB = (A^c \cup B^c)^c$ ($A^c \cup B^c \in \mathcal{F}$ BY E.3; $(A^c \cup B^c)^c \in \mathcal{F}$ BY E.2)

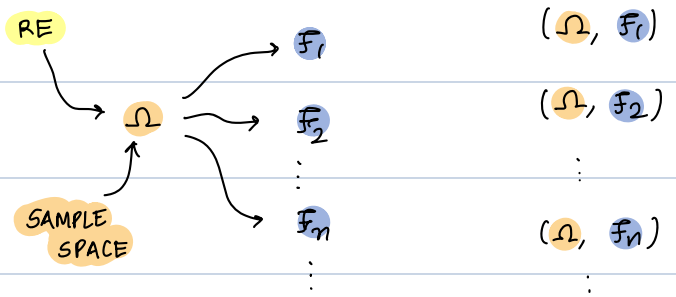
$(A^c \cup B^c) \in \mathcal{F}$ BY E.2

$\Rightarrow AB \in \mathcal{F}$

- PROPERTY E.6 $A_1, A_2, \dots, A_n \in \mathcal{F} \Rightarrow A_1 \cdot A_2 \cdot \dots \cdot A_n \in \mathcal{F}$

PROOF: LEFT AS AN EXERCISE!

- PROBABILITY AXIOMS:



$(\Omega, \mathcal{F}_1)$

$(\Omega, \mathcal{F}_2)$

$(\Omega, \mathcal{F}_n)$

MEASURABLE SPACES

- PROBABILITY MEASURE SPACE

$(\Omega, \mathcal{F}) \rightsquigarrow (\Omega, \mathcal{F}, P)$ ← PROBABILITY MEASURE SPACE

- P IS A SET FUNCTION : $P: \mathcal{F} \to \mathbb{R}$ OR $P: \mathcal{F} \to [0,1]$
  - PROBILITY / PROBABILITY MEASURE

- AXIOM P.1 : $A \in \mathcal{F} \Rightarrow P(A) \geq 0$.

- AXIOM P.2 : $A, B \in \mathcal{F}$ and A and B are mutually exclusive (disjoint),

  i.e. $AB = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$.

- AXIOM P.3. $P(\Omega) = 1$.

- PROPERTY P.4 $A \in \mathcal{F} \Rightarrow P(A^c) = 1 - P(A)$

- PROOF : $A, A^c \in \mathcal{F}$, $A \cap A^c = \phi$, BY P.2 $\Rightarrow P(\overbrace{A \cup A^c}^{\Omega}) = P(A) + P(A^c)$

  $\Rightarrow P(\Omega) = P(A) + P(A^c)$

  $\Rightarrow$ BY P.3 : $1 = P(A) + P(A^c)$

  $\Rightarrow P(A^c) = 1 - P(A)$

→ A QUICK REVIEW

RE : TOSS A COIN

$\Omega = \{H, T\}$

$\mathcal{F}$ = A COLLECTION OF SUBSETS OF Ω THAT OBEYS E.1 - E.3

$\mathcal{F}_1 = \{\phi, \Omega\}$ $\mathcal{F}_2 = \{\phi, \{H\}, \{T\}, \{H, T\} = \Omega\}$

$(\Omega, \mathcal{F}_1)$ and $(\Omega, \mathcal{F}_2)$ ARE TWO MEASURABLE SPACES.

DEFINE $P_1 : \mathcal{F}_1 \longrightarrow \mathbb{R}$

$P_1(\phi) = 0$

$P_1(\Omega) = 1$

$\Rightarrow$ $(\Omega, \mathcal{F}_1, P_1)$ IS A PROBABILITY MEASURE SPACE.

- DEFINE $P_{21} : \mathcal{F}_2 \longrightarrow \mathbb{R}$

$P_{21}(\phi) = 0$ $\quad P_{21}(\{H\}) = \frac{1}{2}$

$P_{21}(\Omega) = 1$ $\quad P_{21}(\{T\}) = \frac{1}{2}$

$\Rightarrow$ $(\Omega, \mathcal{F}_2, P_{21})$ IS A PROBABILITY MEASURE SPACE.

- DEFINE $P_{22} : \mathcal{F}_2 \longrightarrow \mathbb{R}$

$P_{22}(\phi) = 0$ $\quad P_{22}(\{H\}) = p$ $\quad p \in [0,1]$

$P_{22}(\Omega) = 1$ $\quad P_{22}(\{T\}) = 1-p$

$\Rightarrow$ $(\Omega, \mathcal{F}_2, P_{21})$ IS A PROBABILITY MEASURE SPACE.

CONSIDER RE_2 : ROLLING A DIE

$\Omega_2 = \{1, 2, 3, 4, 5, 6\}$ $\quad \mathcal{F}_1 = \{\phi, \Omega\}$ ← FROM TOSSING A COIN

$(\Omega_2, \mathcal{F}_1)$ ← NOT A VALID MEASURABLE SPACE ; $(\Omega_2, \mathcal{F}_2)$ ← VALID MEASURABLE SPACE $\quad \mathcal{F}_2 = \{\phi, \Omega_2\}$ ← FROM ROLLING A DIE

- PROPERTY p.5. $\forall A \in \mathcal{F}$, $P(A) \leq 1$

PROOF : $P(A) = 1 - P(A^c)$ p.4

$P(A) \leq 1$ p.1 : $P(A^c) \geq 0$

• PROPERTY p.6. $P(\phi) = 0$

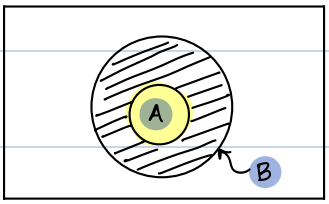
PROOF : $P(A) = 1 - P(A^C)$ p.4

SET $A = \Omega \Rightarrow P(\Omega) = 1 - P(\phi)$

$\Rightarrow P(\phi) = 0$ p.3

• PROPERTY p.7. $A \subset B \Rightarrow P(A) \leq P(B)$ MONOTONICITY OF P

PROOF :



/// : BA^C

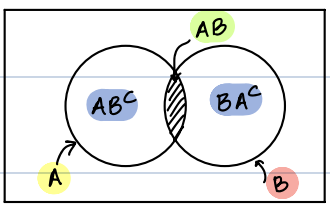
$B = A \cup B = A \cup BA^C$

NOT DISJOINT DISJOINT

$\Rightarrow P(B) = P(A) + P(BA^C)$ p.2.

$\Rightarrow P(B) \geq P(A)$ p.1. $P(BA^C) \geq 0$

• PROPERTY p.8. $P(A \cup B) = P(A) + P(B) - P(AB)$



$A \cup B = AB^C \cup BA^C \cup AB$

DISJOINT

$\Rightarrow P(A \cup B) = P(AB^C) + P(BA^C) + P(AB)$ p.2

$\Rightarrow P(A \cup B) = P(AB^C) + P(BA^C) + P(AB) + P(AB) - P(AB)$

$= P(AB^C) + P(AB) + P(BA^C) + P(AB) - P(AB)$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(AB)$: ADDITION THEOREM

- PROPERTY P.9. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$

PROOF : EXERCISE