ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 29 : BINARY HYPOTHESIS TESTING FOR CONTINUOUS TYPE RANDOM VARIABLES

- TOPICS TO COVER (BASED ON CH 3.10)
  - → BINARY HYPOTHESIS TESTING FOR CONTINUOUS TYPE RANDOM VARIABLES

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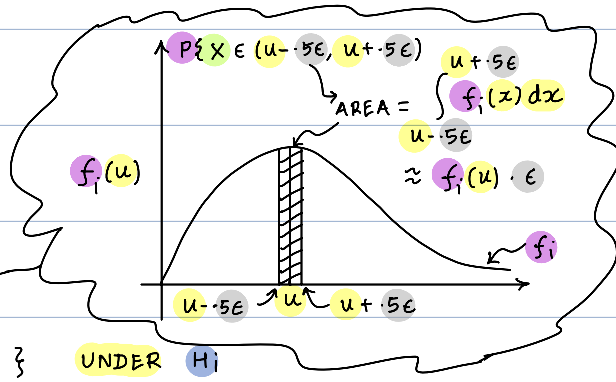
SUPPOSE f_1 AND f_0 ARE TWO KNOWN PDFS.

IF H_i IS TRUE, X (CONT.-TYPE) $\sim f_i$ $i = 0$ OR 1

GOAL : GIVEN A RANDOM OBSERVATION X, WE NEED TO DEVELOP A DECISION RULE TO SPECIFY WHICH HYPOTHESIS (H_1 OR H_0) TO DECLARE!

NOTE THAT FOR X CONT.-TYPE $P\{X = u\} = 0 \quad \forall\, u \in \mathbb{R}$ UNDER EITHER H_i



HOWEVER, WE KNOW THAT, IF ϵ IS SMALL

$$f_i(u)\,\epsilon \approx P\left\{X \in \left(u - \frac{\epsilon}{2},\, u + \frac{\epsilon}{2}\right)\right\} \text{ UNDER } H_i$$

$\therefore$ WE STILL CALL $f_i(u)$: LIKELIHOOD OF $X = u$ UNDER H_i

FOLLOWING THE IDEA SIMILAR TO DISCRETE-TYPE RVS, WE DEFINE

LIKELIHOOD RATIO FOR u IN SUPPORT OF f_1 OR f_0 AS

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)}$$

A LIKELIHOOD RATIO TEST (LRT) WITH THRESHOLD τ IS DEFINED BY

$$\Lambda(X) \begin{cases} > \tau & \text{DECLARE } H_1 \text{ IS TRUE} \\ < \tau & \text{" } H_0 \text{ " "} \end{cases}$$

WHEN $\tau = 1$, WE GET THE MAXIMUM LIKELIHOOD (ML) TEST

ML TEST: $$\Lambda(X) \begin{cases} > 1 & \text{DECLARE } H_1 \text{ IS TRUE} \\ < 1 & \text{" } H_0 \text{ " "} \end{cases}$$

WHEN $\tau = \frac{\pi_0}{\pi_1}$, WHERE π_i IS THE PRIOR PROBABILITY H_i IS TRUE $i = 0, 1$,
WE GET THE MAXIMUM A POSTERIORI (MAP) TEST

MAP TEST: $$\Lambda(X) \begin{cases} > \pi_0/\pi_1 & \text{DECLARE } H_1 \text{ IS TRUE} \\ < \pi_0/\pi_1 & \text{" } H_0 \text{ " "} \end{cases}$$

FURTHERMORE,

$$p_{\text{FALSE-ALARM}} = P\{\text{DECLARE } H_1 \text{ TRUE} \mid H_0\}$$

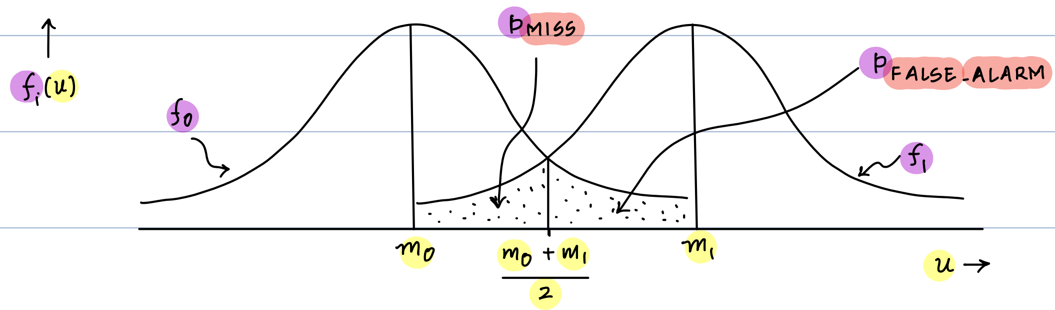
$$p_{\text{MISS}} = P\{\text{DECLARE } H_0 \text{ TRUE} \mid H_1\}$$

AVERAGE ERROR PROB.: $p_e = \pi_0\, p_{\text{FALSE-ALARM}} + \pi_1\, p_{\text{MISS}}$

EXAMPLE: $\quad H_i : \quad X \sim N(m_i, \sigma^2) \quad i = 0, 1 \quad \sigma > 0 \quad m_0 < m_1$

IDENTIFY THE ML AND MAP DECISION RULES, AND CALCULATE THE PROBS OF FALSE ALARM AND MISS.

SOLUTION: WE FIRST DISCUSS THE INTUITION AS FOLLOWS:



GIVEN $X = u$ CHECK IF ITS CLOSER TO m_0 OR m_1, i.e.,

$$u > \frac{m_0 + m_1}{2} \quad \text{DECLARE } H_1$$

$$\text{ELSE} \quad \text{" } H_0$$

E.G. IF $X = 3.2$, $m_0 = 3$ AND $m_1 = 5$: DECLARE f_0 / H_0

WE NEXT OBTAIN THE LRT AS FOLLOWS:

$$f_i(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m_i}{\sigma}\right)^2} \; ; \quad u \in \mathbb{R},\ m_i \in \mathbb{R},\ \sigma > 0$$

$\therefore$ LIKELIHOOD RATIO :
$$\Lambda(u) = \frac{f_1(u)}{f_0(u)}$$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m_1}{\sigma}\right)^2}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-m_0}{\sigma}\right)^2}}$$

$$= e^{\left\{ -\frac{1}{2}\left(\frac{u-m_1}{\sigma}\right)^2 + \frac{1}{2}\left(\frac{u-m_0}{\sigma}\right)^2 \right\}}$$

$(a-b)^2 = a^2 + b^2 - 2ab$

$$= e^{\left\{ \frac{-(u^2+m_1^2-2um_1) + (u^2+m_0^2-2um_0)}{2\sigma^2} \right\}}$$

$$= e^{\left\{ \frac{m_0^2 - m_1^2 + 2u(m_1 - m_0)}{2\sigma^2} \right\}}$$

$a^2 - b^2 = (a+b)(a-b)$

$$= e^{\left\{ \frac{(m_0 - m_1)(m_0 + m_1) + 2u(m_1 - m_0)}{2\sigma^2} \right\}}$$

$$= e^{\left\{ \frac{-(m_1 - m_0)(m_0 + m_1) + 2u(m_1 - m_0)}{2\sigma^2} \right\}}$$

$$= e^{\left\{ \frac{(m_1 - m_0)(2u - (m_0 + m_1))}{2\sigma^2} \right\}}$$

$$\Rightarrow \quad \Lambda(u) = e^{\left\{ \left(u - \frac{(m_0+m_1)}{2}\right)\left(\frac{m_1 - m_0}{\sigma^2}\right) \right\}} \quad \cdots (*)$$

NOTE THAT $\Lambda(u) > 1$ IF AND ONLY IF $u - \frac{(m_0+m_1)}{2} \geq 0 \iff u \geq \frac{(m_0+m_1)}{2}$

$\therefore$ ML TEST :

$$x \begin{cases} > \gamma_{ML} & \text{DECLARE } H_1 \text{ IS TRUE} \\ < \gamma_{ML} & \text{"} \quad H_0 \text{ " "} \end{cases}$$

WHERE $\gamma_{ML} = \frac{m_0 + m_1}{2}$

RECALL THE LRT FOR A GENERAL THRESHOLD τ :

$$\Lambda(X) \begin{cases} > \tau & \text{DECLARE } H_1 \text{ IS TRUE} \\ < \tau & \text{" } H_0 \text{ " "} \end{cases}$$

WHICH IS EQUIVALENT TO

$$\ln \Lambda(X) \begin{cases} > \ln \tau & \text{DECLARE } H_1 \text{ IS TRUE} \\ < \ln \tau & \text{" } H_0 \text{ " "} \end{cases} \quad \because \ln \text{ IS AN INCREASING FUNCTION}$$

(*) $\Rightarrow$ FOR THIS EXAMPLE :

$$\ln \Lambda(u) = \left\{ \left(u - \frac{(m_0 + m_1)}{2} \right) \left(\frac{m_1 - m_0}{\sigma^2} \right) \right\} > \ln \tau$$

$$\Rightarrow \quad u - \frac{m_0 + m_1}{2} > \left(\frac{\sigma^2}{m_1 - m_0} \right) \ln \tau$$

$$\Rightarrow \quad u > \left(\frac{\sigma^2}{m_1 - m_0} \right) \ln \tau + \frac{m_0 + m_1}{2}$$

$\therefore$ GENERAL LRT FOR THIS EXAMPLE :

$$X \begin{cases} > \left(\frac{\sigma^2}{m_1 - m_0} \right) \ln \tau + \frac{m_0 + m_1}{2} & \text{DECLARE } H_1 \text{ IS TRUE} \\ < \quad " & \text{" } H_0 \text{ " "} \end{cases}$$

IN PARTICULAR , THE MAP RULE IS OBTAINED BY SETTING $\tau = \frac{\pi_0}{\pi_1}$

$$x \begin{cases} > \\ < \end{cases} \overbrace{\left(\frac{\sigma^2}{m_1 - m_0}\right) \ln \frac{\pi_0}{\pi_1} + \underbrace{\frac{m_0 + m_1}{2}}_{\gamma_{ML}}}^{\gamma_{MAP}} \quad \begin{matrix} \text{DECLARE } H_1 \text{ IS TRUE} \\ \text{"} \quad H_0 \text{ " "} \end{matrix}$$

WE NEXT CALCULATE THE PROBS. OF FALSE-ALARM AND MISS FOR BOTH ML AND MAP RULES AS FOLLOWS:

ML TEST:

$$P_{FALSE\text{-}ALARM} = P\{\text{DECLARE } H_1 \text{ TRUE} \mid H_0\}$$

$N(m_0, \sigma^2)$

$$= P\{x > \gamma_{ML} \mid H_0\}$$

$N(0, 1)$

$$= P\left\{\frac{x - m_0}{\sigma} > \frac{\gamma_{ML} - m_0}{\sigma}\right\}$$

$$= Q\left(\frac{\gamma_{ML} - m_0}{\sigma}\right)$$

SIMILARLY: MAP TEST: $P_{FALSE\text{-}ALARM} = Q\left(\frac{\gamma_{MAP} - m_0}{\sigma}\right)$

ML TEST:

$$P_{MISS} = P\{\text{DECLARE } H_0 \text{ TRUE} \mid H_1\}$$

$N(m_1, \sigma^2)$

$$= P\{x < \gamma_{ML} \mid H_1\}$$

$N(0, 1)$

$$= P\left\{\frac{x - m_1}{\sigma} < \frac{\gamma_{ML} - m_1}{\sigma}\right\}$$

$$= \Phi\left(\frac{\gamma_{ML} - m_1}{\sigma}\right) = Q\left(\frac{m_1 - \gamma_{ML}}{\sigma}\right)$$

SIMILARLY: MAP TEST: $P_{FALSE\text{-}ALARM} = \Phi\left(\frac{\gamma_{MAP} - m_1}{\sigma}\right) = Q\left(\frac{m_1 - \gamma_{MAP}}{\sigma}\right)$

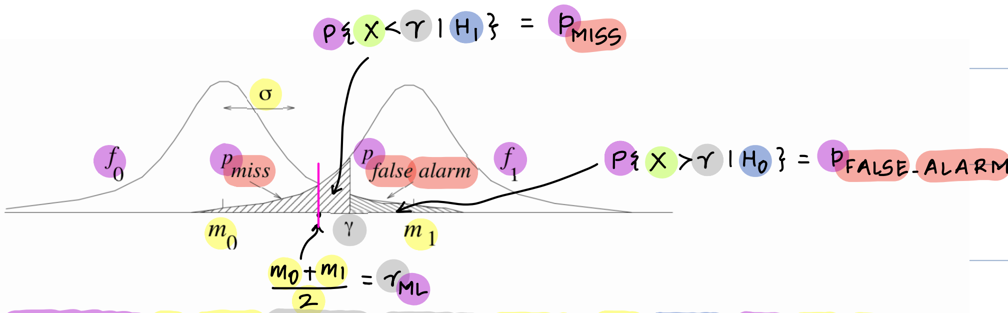


Figure 3.26: Error probabilities for direct threshold detection between two normal pdfs with the same variance.

AVERAGE ERROR PROB. : $p_e = \pi_0 \, p_{FALSE-ALARM} + \pi_1 \, p_{MISS}$

EXERCISE : FOR THIS EXAMPLE, FOR ML TEST, SHOW THAT (UNDER $\pi_0 = \pi_1$)

$$p_{FALSE-ALARM} = p_{MISS} = p_e$$