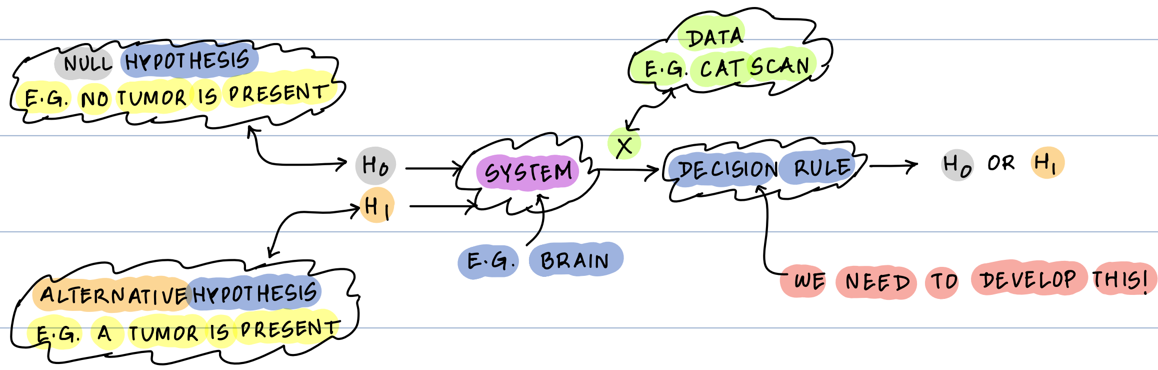
ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 19: BINARY HYPOTHESIS TESTING

- TOPICS TO COVER (BASED ON CH 2.11)

→ MAXIMUM A POSTERIORI (MAP) DECISION RULE

→ MAXIMUM A POSTERIORI (MAP) DECISION RULE



H_0: NULL HYPOTHESIS : IF H_0 IS TRUE, THE PMF OF X IS p_0

H_1: ALTERNATIVE HYPOTHESIS : IF H_1 IS TRUE, THE PMF OF X IS p_1

LIKELIHOOD MATRIX:

E.G. $P(\{X = x \mid H_i\})$, $i = 0, 1$

CONDITIONAL PROB.
'LIKELIHOOD'

	$X=0$	$X=1$	$X=2$	$X=3$	
H_1	0.0	0.1	0.3	0.6	$\rightsquigarrow p_1$
H_0	0.4	0.3	0.2	0.1	$\rightsquigarrow p_0$

PRIOR BELIEFS: $P(H_1) = \pi_1$

E.G. EXPERT KNOWLEDGE $P(H_0) = \pi_0$

$\left.\begin{matrix} P(H_1) = \pi_1 \\ P(H_0) = \pi_0 \end{matrix}\right\}$ PRIOR PROB.

HOW DO YOU USE THE PRIOR BELIEF TO REFINE YOUR DECISION ?

LET'S CONSIDER : $\underbrace{P(\overset{\text{BELIEF}}{H_i} \mid \overset{\text{DATA}}{X = x})}_{\text{UPDATED BELIEF}}$ $i = 0, 1$

POSTERIOR PROB.

DECISION RULE : $D(X)$: DECISION RULE / TEST IS A FUNCTION OF X ← DATA

E.G. DECISION RULE : DECLARE H_1 IF $\overbrace{P(H_1 \mid X = x) > P(H_0 \mid X = x)}^{\text{MAXIMUM A POSETERIORI (MAP) DECISION RULE}}$

LET'S LOOK AT :

$$\underset{\text{UPDATED BELIEF}}{P(H_i \mid X = x)} = \left(\underbrace{P(H_i)}_{\pi_i} \cdot \underbrace{P(X = x \mid H_i)}_{p_i(x)} \right) / P(X = x) \quad \text{'BAYES RULE'}$$

PRIOR ⇝ π_i $\quad$ $p_i(x)$ ← DATA

WE WANT TO COMPARE $P(H_1 \mid X = x)$ AND $P(H_0 \mid X = x)$

$$P(H_1 \mid X = x) = \left(P(H_1) \cdot P(X = x \mid H_1) \right) / P(X = x)$$

$$P(H_0 \mid X = x) = \left(P(H_0) \cdot P(X = x \mid H_0) \right) / P(X = x)$$

COMPARING $P(H_1 \mid X = x)$ AND $P(H_0 \mid X = x)$ IS SAME AS COMPARING :

$$\underbrace{P(H_1)\cdot P(X=x \mid H_1)}_{P(H_1 \,\cap\, X=x)} \quad \text{AND} \quad \underbrace{P(H_0)\cdot P(X=x \mid H_0)}_{P(H_0 \,\cap\, X=x)}$$

JOINT PROBS.

E.G. $P(H_1) = \pi_1 = 0.2$ $\quad P(H_0) = \pi_0 = 0.8$

$P(X=x \mid H_1)$	$X=0$	$X=1$	$X=2$	$X=3$	
H_1	0.0	0.1	0.3	0.6	$\rightsquigarrow$ SUM $= 1$
H_0	0.4	0.3	0.2	0.1	$\rightsquigarrow$ SUM $= 1$

$P(X=x \cap H_1)$	$X=0$	$X=1$	$X=2$	$X=3$	
H_1	$(0.0)\pi_1$	$(0.1)\pi_1$	$(0.3)\pi_1$	$(0.6)\pi_1$	$\rightsquigarrow$ SUM $= \pi_1$
H_0	$(0.4)\pi_0$	$(0.3)\pi_0$	$(0.2)\pi_0$	$(0.1)\pi_0$	$\rightsquigarrow$ SUM $= \pi_0$

$P(X=x \cap H_1)$	$X=0$	$X=1$	$X=2$	$X=3$
H_1	$(0.4)(0.2)$	$(0.3)(0.2)$	$(0.2)(0.2)$	$(0.1)(0.2)$
H_0	$(0.0)(0.8)$	$(0.1)(0.8)$	$(0.3)(0.8)$	$(0.6)(0.8)$

DECISION : H_0 (for $X=0$, $X=1$, $X=2$) $\qquad$ DECISION : H_1 (for $X=3$)

$P(X=x \cap H_1)$	$X=0$	$X=1$	$X=2$	$X=3$	
H_1	0.00	0.02	0.06	0.12	$\rightsquigarrow$ SUM $= 0.2$
H_0	0.32	0.24	0.16	0.08	$\rightsquigarrow$ SUM $= 0.8$

miss

false alarm

A GOOD DECISION RULE / TEST SHOULD MINIMIZE $p_{\text{false alarm}}$ AND p_{miss} !

NOTE THAT YOU CANNOT MINIMIZE BOTH AT THE SAME TIME.

DECISION RULE : GIVEN $X = x$, COMPARE $p_1(x)\,\pi_1$ AND $p_0(x)\,\pi_0$

IF $\pi_1\, p_1(x) > \pi_0\, p_0(x)$: DECLARE H_1

OTHERWISE : H_0

ANOTHER WAY : IF $\dfrac{\pi_1}{\pi_0}\,\dfrac{p_1(x)}{p_0(x)} > 1$: DECLARE H_1

OTHERWISE : H_0

DEFINE $\Lambda(X=x) = \dfrac{p_1(x)}{p_0(x)}$ ← LIKELIHOOD RATIO

CONDITIONAL LIKELIHOODS

IF $\Lambda(X=x) > \dfrac{\pi_0}{\pi_1}$: DECLARE H_1

OTHERWISE : H_0

τ IS A USER-DEFINED THRESHOLD

IN GENERAL : IF $\Lambda(X=x) > \tau$: DECLARE H_1

OTHERWISE : H_0

AVERAGE ERROR OF A TEST IN GENARAL :

COND. PROBS

$p_e = \pi_0\, p_{\text{false-alarm}} + \pi_1\, p_{\text{miss}}$

: SUM OF ALL UNUNDERLINED VALUES IN THE 'JOINT' PROB. MATRIX