ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 14 : NEGATIVE BINOMIAL AND POISSON DISTRIBUTIONS

- TOPICS TO COVER (BASED ON CH 2.6 – 2.7)
  - → NEGATIVE BINOMIAL DISTRIBUTION
  - → POISSON DISTRIBUTION

→ NEGATIVE BINOMIAL DISTRIBUTION

- SEQUENCE OF INDEP. BERNOULLI TRIALS WITH PARAMETER p

S_r := # OF TRIALS REQUIRED FOR r SUCCESSES (1)

POSSIBLE VALUES OF S_r = $r, \; r+1, \; r+2, \; \ldots$

$\{S_r = n\}$:= n TRIALS NEEDED FOR FIRST r SUCCESSES

$$n \geq r \qquad k = n - r$$

$$\{S_r = n\} := \underbrace{\overbrace{\_\ \ \_\ \ \_\ \ \cdots}^{r-1 \text{ 1s and } k = n-r \text{ zeros}}}_{n-1 \text{ TRIALS}} \;\; \underline{1}$$

TRIAL # : 1 2 3 … n

$$p_{S_r}(n) = P\{S_r = n\} := \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \cdot p, \qquad n \geq r$$

$$S_r \sim NB(r, p)$$

IF $r = 1$: S_1 := # OF TRIALS NEEDED FOR FIRST '1' SUCCESS

'GEOMETRIC DIST.'

$$P\{S_1 = n\} := 1 \cdot p^0 (1-p)^{n-1} p$$

$$= (1-p)^{n-1} p \qquad n \geq r = 1 \ \therefore \ n \geq 1$$

PMF VERIFICATION :

- $p_{S_r}(n) \geq 0$
- $\sum_{n=r}^{\infty} p_{S_r}(n) = 1$

WE NEED TO SHOW!

$$\sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r (1-p)^{n-r} = 1$$

CONSIDER:

NEG. EXPONENT

$$(1-x)^{-r} := \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} x^k \quad : \text{NEG. BINOMIAL EXPANSION}$$

TAKE $x = 1-p$ and SET $k = n-r \Rightarrow r = n-k \Rightarrow$ $k=0 \longrightarrow n=r$, $k=\infty \longrightarrow n=\infty$

$$(1-(1-p))^{-r} = \sum_{n=r}^{\infty} \binom{n-1}{r-1} (1-p)^{n-r}$$

$$\Rightarrow \quad p^{-r} = \sum_{n=r}^{\infty} \binom{n-1}{r-1} (1-p)^{n-r}$$

$$\Rightarrow \quad 1 = \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$1 = \sum_{n=r}^{\infty} \text{PMF OF } NB(r, p)$$

· MEAN AND VARIANCE OF $S_r \sim NB(r, p)$

OBSERVE THAT $S_r = L_1 + L_2 + \cdots + L_r$ where $L_i \sim$ GEOMETRIC(p) $\forall\, i = 1, \ldots, r$

INDEP.

TRIALS TO GET r SUCCESSES

r TIMES TRIALS TO GET 1 SUCCESS

$$E(S_r) = E(L_1 + \cdots + L_r)$$

INDEP. OF $L_1, \ldots, L_n$ IS NOT NEEDED

LINERAITY OF EXP.

$$= E(L_1) + \cdots + E(L_r)$$

$$E(S_r) = r \cdot \frac{1}{p}$$

MEAN OF A GEOMETRIC RV

INDEP. OF $L_1, \ldots, L_n$ IS NEEDED

WE WILL REVISIT THIS!

WE CAN ALSO CALCULATE: $Var(S_r) := Var(L_1) + \cdots + Var(L_r)$

$$= r \cdot \frac{(1-p)}{p^2}$$

VARIANCE OF A GEOMETRIC RV

→ POISSON DISTRIBUTION

· n INEDEP. BERNOULLI TRIALS

BINOMIAL (n, p) : # OF SUCCESSES IN THOSE n TRIALS

CONSIDER: $p_b(0) = \binom{n}{0} p^0 (1-p)^{n-0}$

BINOMIAL (n, p) RV

$\Rightarrow$ $p_b(0) = (1-p)^n$

$\Rightarrow$ $\ln p_b(0) = \ln (1-p)^n$

$$= n \ln(1-p)$$

$$(*) \Rightarrow \ln p_b(0) = n \ln\left(1 - \frac{\lambda}{n}\right)$$

DEFINE $\underbrace{\lambda := np}_{\text{MEAN OF BINOMIAL}}$ CONDITION 1

WE KNOW THAT: $\ln(1+u) = u + o(u)$ WHERE $\frac{o(u)}{u} \to 0$ AS $u \to 0$.

TAYLOR'S THEOREM

$$\therefore \quad (*) \Rightarrow \ln p_b(0) = n\left(-\frac{\lambda}{n} + o\left(-\frac{\lambda}{n}\right)\right)$$

$$\Rightarrow \lim_{n\to\infty} \ln p_b(0) = \lim_{n\to\infty} n \cdot -\frac{\lambda}{n} + \underbrace{n \cdot o\left(-\frac{\lambda}{n}\right)}_{0}$$

AS $n \to \infty \Rightarrow -\frac{\lambda}{n} \to 0$

CONDITION 2

$$\Rightarrow \lim_{n\to\infty} \ln p_b(0) = \underbrace{\lim_{n\to\infty} -\lambda}_{-\lambda} + \underbrace{\lim_{n\to\infty} n\, o\left(-\frac{\lambda}{n}\right)}_{0}$$

$$\Rightarrow \ln p_b(0) \to -\lambda \quad \text{AS } n \to \infty$$

$$\Rightarrow p_b(0) \to e^{-\lambda} \quad \text{AS } n \to \infty \quad \ldots (**)$$

NOW CONSIDER:

$$p_b(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\ldots(n-k-1)\,(n-k)!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\ldots(n-k-1)}{k!} p^k (1-p)^{n-k}$$

SET $p = \frac{\lambda}{n}$

$$\Rightarrow p_b(k) = \frac{n(n-1)\cdots(n-k-1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{=p_b(0)}\ \underbrace{\frac{n(n-1)\cdots(n-k-1)}{n^k}}_{\downarrow \text{ as } n\to\infty\ 1}\ \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\downarrow \text{ as } n\to\infty\ 1}$$

$\frac{\lambda}{n} \to 0$ as $n\to\infty$

(**)

TAKE $\lim_{n\to\infty}$:

$$\lim_{n\to\infty} p_b(k) = \frac{\lambda^k}{k!} e^{-\lambda}\ \underbrace{\lim_{n\to\infty}\frac{n(n-1)\cdots(n-k-1)}{n^k}}_{1}\ \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k}}_{1}$$

$$\Rightarrow p_b(k) \longrightarrow \frac{\lambda^k e^{-\lambda}}{k!}$$ AS $n\to\infty$ WHERE $\lambda = np$

'p IS SMALL'

DEFINITION. A RV X IS SAID TO HAVE A POISSON DIST. WITH PARAMETER λ IF ITS PMF IS GIVEN AS:

$$p_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!}, & k = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

SUPPORT

PMF VERIFICATION:

- $p_X(k) \geq 0 \quad \forall k$
- $\sum_{k=0}^{\infty} p_X(k) = 1$

$$\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$$

WE KNOW THAT $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^{k}}{k!} = e^{-\lambda} e^{\lambda} = 1$$

MEAN OF X :

$$\begin{aligned}
EX &:= \sum_{k=0}^{\infty} k \, p_X(k) \\
&= \sum_{k=0}^{\infty} k \frac{e^{-\lambda} \lambda^{k}}{k!} \\
&= \sum_{k=0}^{\infty} k \frac{e^{-\lambda} \lambda \lambda^{k-1}}{k(k-1)!} \\
&= \lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} \\
&= \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} = \lambda \sum_{l=0}^{\infty} \frac{e^{-\lambda} \lambda^{l}}{l!} \\
&= \lambda
\end{aligned}$$

VARINANCE OF X :

$$Var(X) := EX^2 - (EX)^2$$

$$\begin{aligned}
E(X^2 - X) &:= \sum_{k=0}^{\infty} (k^2 - k) \, p_X(k) \\
&= \sum_{k=0}^{\infty} (k^2 - k) \frac{e^{-\lambda} \lambda^{k}}{k!} \\
&= \sum_{k=0}^{\infty} (k^2 - k) \frac{e^{-\lambda} \lambda^2 \lambda^{k-2}}{k(k-1)(k-2)!} \\
&= \lambda^2 \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^{k-2}}{(k-2)!} \\
&= \lambda^2 \sum_{l=0}^{\infty} \frac{e^{-\lambda} \lambda^{l}}{l!} \\
&= \lambda^2
\end{aligned}$$

$$\Rightarrow EX^2 - EX = \lambda^2$$

$$\Rightarrow EX^2 - \lambda = \lambda^2 \quad \Rightarrow EX^2 = \lambda^2 + \lambda$$

$$\therefore \quad Var(X) = EX^2 - (EX)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$Var(X) = \lambda$$

ALTERNATIVELY :

$$B \sim \text{BINOMIAL}(n, p) \quad \text{and} \quad X \sim \text{POISSON}(\lambda)$$

$$\lambda = \frac{n}{p} \quad \text{and} \quad n \to \infty$$

$$EB = np = n\frac{\lambda}{n} \longrightarrow \lambda = EX$$

as $n \to \infty$

$$Var\,B = np(1-p) = n\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right) \longrightarrow \lambda = Var(X)$$

as $n \to \infty$