LECTURE 13: GEOMETRIC DISTRIBUTION

- TOPICS TO COVER (BASED ON CH 2.5)

  → GEOMETRIC DISTRIBUTION

GEOMETRIC DISTRIBUTION

- SEQUENCE OF INDEP. BERNOULLI TRIALS WITH $0 \le p \le 1$

ONE WAY TO DEFINE ⇝ X = # OF TRIALS CONDUCTED UNTIL 'BEFORE' THE FIRST SUCCESS

DON'T USE THIS FOR OUR CLASS

E.G.

$X = 1 \qquad P\{X=1\} = \underbrace{(1-p)}_{\text{1 TRIAL BEFORE THE FIRST SUCCESS}}\, p$ ← FIRST SUCCESS

$X = 2 \qquad P\{X=2\} = (1-p)^2\, p$

2 ← 2 TRIALS BEFORE THE FIRST SUCCESS

p ← FIRST SUCCESS

$\vdots \qquad\qquad \vdots$

$X = k \qquad P\{X=k\} = (1-p)^k\, p \qquad k \geq 1$

⇒ PMF OF X :

$$p_X(k) = \begin{cases} (1-p)^k p, & k \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$p_X(k)$ IS A VALID PMF!

- $p_X(k) \geq 0 \ \forall k$: EASY TO VERIFY.
- $\sum_k p_X(k) = 1$

$$\sum_k p_X(k) = \sum_{k=0}^{\infty} (1-p)^k p$$

GEOMETRIC SERIES

$$= p \frac{1}{1-(1-p)} = 1 \qquad \because \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad x \in [0,1]$$

· SEQUENCE OF INDEP. BERNOULLI TRIALS WITH $0 \le p \le 1$

ANOTHER WAY TO DEFINE $\rightsquigarrow X$ = # OF TRIALS CONDUCTED UNTIL THE FIRST SUCCESS INCLUDING THE TRIAL THAT GAVE SUCCESS

USE THIS FOR OUR CLASS

E.G. $X = 1$ $\quad P\{X=1\} = p$ ← FIRST SUCCESS

$X = 2$ $\quad P\{X=2\} = (1-p)^1 \, p$ ← 1 TRIAL BEFORE THE FIRST SUCCESS; ← FIRST SUCCESS

$\vdots \qquad \vdots$

$X = k$ $\quad P\{X=k\} = (1-p)^{k-1} p \qquad k \ge 1$

$\Rightarrow$ PMF OF X :

$$p_X(k) = \begin{cases} (1-p)^{k-1} p, & k \ge 1, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$p_X(k)$ IS A VALID PMF!

- $p_X(k) \ge 0 \;\; \forall k$: EASY TO VERIFY.

- $\sum_k p_X(k) = 1$

$$\sum_k p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= p \frac{1}{1-(1-p)} \qquad \because \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \quad x \in [0,1]$$

GEOMETRIC SERIES

$$= 1$$

· MEAN OF A GEOMETRIC DIST.

$$EX := \sum_k k \, p_X(k)$$

$$= \sum_{k=1}^{\infty} k \, (1-p)^{k-1} \cdot p \qquad \cdots (*)$$

CONSIDER:
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \; ; \quad |x| \in (0,1) \qquad \cdots (i)$$

TAKING DERIVATIVES ON EACH SIDE OF (i) W.R.T. x :

$$\frac{\partial}{\partial x} \sum_{k=0}^{\infty} x^k = \frac{\partial}{\partial x} \frac{1}{1-x} \qquad \because \frac{\partial}{\partial \theta}\theta^n = n\theta^{n-1}$$

$$\sum_{k=0}^{\infty} \frac{\partial}{\partial x} x^k = -1(1-x)^{-2}(-1)$$

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \qquad \cdots (ii)$$

TAKE $x = (1-p)$ in (ii)

$$\sum_{k=0}^{\infty} k(1-p)^{k-1} = \frac{1}{(1-(1-p))^2}$$

$$= \frac{1}{p^2}$$

$$\sum_{k=0}^{\infty} k(1-p)^{k-1} \cdot p = \frac{1}{p} \qquad \cdots (**)$$

COMPARING (*) AND (**),

$$EX = \frac{1}{p}$$

ANOTHER WAY TO COMPUTE EX :

- IF THE OUTCOME OF THE FIRST TRIAL IS 1

PROB. OF 1

$\Rightarrow$ CONDITIONED ON THIS, $EX \mid \; = p \cdot 1$

VALUE OF X : ONE TRIAL

- IF THE OUTCOME OF THE FIRST TRIAL IS 0

$\Rightarrow$ CONDITIONED ON THIS, $EX|$ = $(1-p)(1+E(\tilde{X}))$

\# OF ADDITIONAL TRIALS TO GET THE FIRST SUCCESS

$$\Rightarrow EX = p \cdot 1 + (1-p)(1+E\tilde{X})$$

NOTE THAT X and $\tilde{X}$ HAVE THE SAME DIST.

$$\Rightarrow EX = p + (1-p)(1+EX)$$

$$EX = p + (1-p) + (1-p)EX$$

$$EX = p + (1-p) + EX - pEX$$

$$\Rightarrow pEX = 1 \Rightarrow EX = \frac{1}{p}$$

- VARIANCE OF X :

$$Var(X) := EX^2 - (EX)^2$$

$$\rightarrow EX^2 := \sum_k k^2 p_X(k) = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

RECALL (ii) : $\displaystyle\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$

TAKING DERIVATIVES ON EACH SIDE OF (ii) W.R.T. x :

$$\frac{\partial}{\partial x} \sum_{k=0}^{\infty} k x^{k-1} = \frac{\partial}{\partial x} \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{k=0}^{\infty} k(k-1)x^{k-2} = -2(1-x)^{-2-1}(-1)$$

$$\sum_{k=0}^{\infty} k(k-1)x^{k-2} = \frac{2}{(1-x)^3} \qquad \cdots \text{(ii)}$$

TAKE $x = (1-p)$ in (ii) :

$$\sum_{k=0}^{\infty} k(k-1)(1-p)^{k-2} = \frac{2}{(1-(1-p))^3} = \frac{2}{p^3}$$

$$\sum_{k=0}^{\infty} k^2(1-p)^{k-2} - \sum_{k=0}^{\infty} k(1-p)^{k-2} = \frac{2}{p^3}$$

$$\sum_{k=0}^{\infty} k^2(1-p)^{k-2}p - \sum_{k=0}^{\infty} k(1-p)^{k-2}p = \frac{2}{p^2}$$

$$(1-p)^{-1}\left(\sum_{k=0}^{\infty} k^2(1-p)^{k-1}p - \sum_{k=0}^{\infty} k(1-p)^{k-1}p\right) = \frac{2}{p^2}$$

$$\Rightarrow (1-p)^{-1}(EX^2 - EX) = \frac{2}{p^2} \Rightarrow EX^2 - \frac{1}{p} = \frac{2(1-p)}{p^2}$$

$$\Rightarrow EX^2 = \frac{2(1-p)+p}{p^2}$$

$$= \frac{2-p}{p^2}$$

ANOTHER WAY TO CALCULATE EX^2 SIMILAR TO EX :

SHOW!

$$EX^2 = 1^2 p + (1-p)E(1+X)^2 \Rightarrow EX^2 = \frac{2-p}{p^2}$$

HENCE, $$Var(X) = EX^2 - (EX)^2$$

SHOW!

$$Var(X) = \frac{1-p}{p^2}$$

MEMORYLESS PROPERTY OF GEOMETRIC DISTRIBUTION :

TENNIS SCORING : The player who wins at least 4 points first with a minimum difference of 2 points wins the game !

Let the state of the game is as follows :

SCORE	# OF POINTS
♡	0
15	1
30	2
40	3
45	4

PLAYER 1 : ♡ 15 30

PLAYER 2 : ♡ 15 30 40

: ADVANTAGE PLAYER 2

EQUIVALENT

PLAYER 1 : ♡ 15 30 40

PLAYER 2 : ♡ 15 30 40

: 'DEUCE'

PLAYER 1 : ♡ 15 30

PLAYER 2 : ♡ 15 30 40 45

: PLAYER 2 WINS

PLAYER 1 : ♡ 15 30 40

PLAYER 2 : ♡ 15 30 40 45

: ADVANTAGE PLAYER 2

PLAYER 1 : ♡ 15 30 40 45

PLAYER 2 : ♡ 15 30 40

: ADVANTAGE PLAYER 1

EQUIVALENT

PLAYER 1 : ♡ 15 30 40

PLAYER 2 : ♡ 15 30

: ADVANTAGE PLAYER 1

⇒ A DEUCE / ADVANTAGE IS THE DEUCE / ADVANTAGE IRRESPECTIVE OF WHEN IT ARISES IN THE GAME !

" THE GAME IS MEMORY LESS ! "

WORDS OF WISDOM

WOW ! : THE NATURE MAY NOT REMEMBER YOUR EFFORT !

YOU MAY BE JUST A STEP AWAY FROM SUCCESS !

SO KEEP GOING !

FOR A GEOMETRIC RANDOM VARIABLE $X \sim \text{GEOMETRIC}(p)$:

$$P(X > k+n \mid \underbrace{X > n}_{\text{ALREADY HAPPENED}}) = P(X > k) \qquad k, n \geq 1$$

WE NEED TO SHOW THIS!

PROOF : TAKE :

$$P(X > k+n \mid X > n) \overset{\text{DEF. OF COND. PROB.}}{=} \frac{P(X > k+n \cap X > n)}{P(X > n)}$$

$$X > k+n \Rightarrow X > n \quad : \quad X > k+n \subset X > n$$

$$\Rightarrow P(X > k+n \mid X > n) = \frac{P(X > k+n)}{P(X > n)} \quad \ldots (*)$$

NOW CONSIDER :

$$P(X > x) = \sum_{l=x+1}^{\infty} (1-p)^{l-1} p$$

$$= p \left\{ (1-p)^{x+1-1} + (1-p)^{x+2-1} + (1-p)^{x+3-1} + \cdots \right\}$$

$$= p \left\{ (1-p)^{x} + (1-p)^{x+1} + (1-p)^{x+2} + \cdots \right\}$$

$$= p \, \frac{(1-p)^{x}}{1-(1-p)}$$

$$\Rightarrow P(X > x) = (1-p)^{x} \quad \ldots (**)$$

USING $(**)$ IN $(*)$:

$$P(X > k+n \mid X > n) = \frac{(1-p)^{k+n}}{(1-p)^{k}}$$

$$= (1-p)^{n}$$

$$= P(X > n)$$

HENCE THE RESULT!