LECTURE 12: BINOMIAL DISTRIBUTION

- TOPICS TO COVER (BASED ON CH 2.4)

→ BINOMIAL DISTRIBUTION

BINOMIAL DISTRIBUTION

n INDEPENDENT BERNOULLI TRIALS, EACH RESULTING IN A ONE w.p. p ; $0 \le p \le 1$

DEFINE X = # OF SUCCESSES IN n INDEPENDENT BERNOULLI TRIALS

OBSERVE : $0 \le X \le n$

EVENT: $\{X = k\}$ $\quad 0 \le k \le n$

k SUCCESSES IN n INDEP. TRIALS

E.G. GETTING k HEADS IN n INDEP. COIN TOSSES

PMF OF X

$$p_X(k) := P\{X = k\} = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & \underbrace{0 \le k \le n}_{\text{SUPPORT OF } X} \\ 0, & \text{otherwise.} \end{cases} \quad , \quad 0 \le p \le 1$$

PMF VERIFICATION:

- $p_X(k) \geq 0 \quad \forall \; k$: EASY TO VERIFY!

$$\sum_{k} p_X(k) = 1$$

WE NEED TO SHOW!

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} + 0 = 1$$

LET'S CONSIDER:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= \binom{2}{0} b^0 a^{2-0} \quad \binom{2}{1} b^1 a^{2-1} + \binom{2}{2} b^2 a^{2-2}$$

$$(a+b)^3 = (a+b)(a+b)^2$$

$$= (a+b)(a^2 + 2ab + b^2)$$

$$= a^3 + 2a^2 b + ab^2 + ba^2 + 2ab^2 + b^3$$

$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$= \binom{3}{0} b^0 a^{3-0} + \binom{3}{1} b^1 a^{3-1} + \binom{3}{2} b^2 a^{3-2} + \binom{3}{3} b^3 a^{3-3}$$

IN GENERAL:

$$(a+b)^n = \binom{n}{0} b^0 a^{n-0} + \binom{n}{1} b^1 a^{n-1} + \cdots + \binom{n}{n} b^n a^{n-n}$$

$$(*) \qquad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k}$$: BINOMIAL EXPANSION

TAKE $b = p$ and $a = (1-p)$; $0 \le p \le 1$ IN $(*)$

$$(p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$1^n = 1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

HENCE $p_X(k)$ IS A VALID PMF!

EXPECTATION AND VARIANCE OF X:

- $$EX := \sum_k k\, p_X(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} + \underbrace{0 \cdot \binom{n}{0} p^0 (1-p)^{n-0}}_{0}$$

$$= \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \frac{n(n-1)!}{k(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$\hookrightarrow \binom{n-1}{k-1}$ NOTE: $(n-k) = (n-1) - (k-1)$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)}$$

$$= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} (1-p)^{n-1-l} \qquad l = k-1$$

$$= np \underbrace{\sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} (1-p)^{n-1-l}}_{\text{SUM OF A BINOMIAL PMF WITH } n-1 \text{ TRIALS}}$$

$$= np \cdot 1$$

$$\Rightarrow EX = np$$

- VARIANCE OF X:

$$Var(X) = EX^2 - (EX)^2$$

- $$EX^2 := \sum_k k^2 p_X(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} + \underbrace{0^2 \cdot \binom{n}{0} p^0 (1-p)^{n-0}}_{0}$$

... (EXERCISE!)

$$= n^2p^2 + np(1-p)$$

$$\therefore \quad Var(X) = n^2p^2 + np(1-p) - (np)^2$$

$$= np(1-p)$$

MODE AND MEDIAN OF X :

- REFER TO THE BOOK (PROF. HAJEK'S NOTES)