ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 11: INDEPENDENCE OF RANDOM VARIABLES AND BERNOULLI DISTRIBUTION

- TOPICS TO COVER (BASED ON CH 2.4)
  - → INDEPENDENCE OF RANDOM VARIABLES
  - → BERNOULLI DISTRIBUTION

INDEPENDENCE OF RANDOM VARIABLES

DEFINITION. RANDOM VARIABLES X and Y ARE INDEP. IF ANY EVENT OF THE FORM $X \in A$ IS INDEP. OF ANY EVENT OF THE FORM $X \in B$.

$$X \in A \quad : \quad \{\omega : X(\omega) \in A\} \in \mathcal{F}$$

$$Y \in B \quad : \quad \{\omega : Y(\omega) \in B\} \in \mathcal{F}$$

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \qquad \forall \; A, B \subseteq \mathbb{R}$$

PROPERTY : IF X and Y ARE INDEP. THEN

$$\underbrace{P\{X = i, Y = j\}}_{\text{JOINT PMF : } p_{XY}(i,j)} = \underbrace{P\{X = i\}}_{p_X(i)} \cdot \underbrace{P\{Y = j\}}_{p_Y(j)} \qquad i, j \in \mathbb{R}$$

PMFS OF X and Y

PROOF: GIVEN THAT X AND Y ARE INDEP.

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \quad \forall \; A, B \subseteq \mathbb{R}$$

TAKE $A = \{i\}$, $B = \{j\}$ $\quad i, j \in \mathbb{R}$

$$P\{X = i, Y = j\} = P\{X = i\} \cdot P\{Y = j\}$$

PROPERTY: IF $\underbrace{P\{X = i, Y = j\}}_{p_{XY}(i,j)} = \underbrace{P\{X = i\}}_{p_X(i)} \cdot \underbrace{P\{Y = j\}}_{p_Y(j)}$ $\quad i, j \in \mathbb{R}$

JOINT PMF: $p_{XY}(i,j)$

PMFS OF X and Y

THEN X and Y ARE INDEP.

PROOF: WE NEED TO SHOW:

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \quad \forall \; A, B \subseteq \mathbb{R}$$

CONSIDER:

$$P\{X \in A, Y \in B\} \overset{\text{AXIOM P.2}}{=} \sum_{(i,j):\, i \in A,\, j \in B} P\{X = i, Y = j\}$$

GIVEN

$$= \sum_{(i,j):\, i \in A,\, j \in B} P\{X = i\} P\{Y = j\}$$

$$= \sum_{i:\, i \in A} P\{X = i\} \cdot \sum_{j:\, j \in B} P\{Y = j\}$$

$$\Rightarrow P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{X \in B\}$$

BERNOULLI DISTRIBUTION

RE: AN EXP. WITH TWO COMPLEMETARY OUTCOMES : BERNOULLI TRIAL

E.G. TOSS A COIN $\Rightarrow$ $\Omega = \{H, T\}$ $\quad P\{H\} = 1 - P\{T\}$

DEFINE $\quad X = \begin{cases} 1 & \text{OUTCOME 1, E.G. H} \\ 0 & \text{OUTCOME 2, E.G. T} \end{cases}$

$$P\{X = 1\} = 1 - P\{X = 0\}$$

TAKE PARAMETER p; $0 \le p \le 1$:

DEFINE $\quad P\{X = 1\} = p \quad$ THEN $\quad P\{X = 0\} = 1 - p$

IN GENERAL, THE PMF CAN BE WRITTEN AS:

$$p_X(x) := P\{X = x\} = \begin{cases} p^x (1-p)^{1-x}, & x = 0, 1, \\ 0, & \text{otherwise.} \end{cases} \qquad 0 \le p \le 1$$

LET'S VERIFY THAT $p_X(x)$ IS INDEED A PMF!

- $p_X(x) \geq 0$: EASY TO VERIFY
- $\sum_x p_X(x) = 1$

$$\sum_{x \in \{0,1\}} p^x (1-p)^{1-x} + 0 = p + (1-p) = 1$$

LET'S CALCULATE THE MEAN AND VARIANCE OF X:

- $EX := \sum_x x\, p_X(x) = \sum_{x \in \{0,1\}} x\, p^x (1-p)^{1-x} + 0$

DEFINITION

$$= p$$

- $\mathrm{Var}(X) := \mathbb{E}X^2 - (\mathbb{E}X)^2$

$$\mathbb{E}X^2 = \sum_x x^2 p_X(x) = \sum_{x \in \{0,1\}} x^2 p^x (1-p)^{1-x} + 0$$

$$= p$$

YOU COULD ALSO NOTE THAT $X^2 = X \Rightarrow \mathbb{E}X^2 = \mathbb{E}X$

$$\Rightarrow \mathrm{Var}(X) = p - p^2$$

$$= p(1-p)$$

→ FOR WHAT VALUE OF p, $\mathrm{Var}(X)$ is maximized ?

FIND $\arg\max_p \mathrm{Var}(X)$

$$\frac{\partial}{\partial p} \mathrm{Var}(X) = 0$$

$$\frac{\partial}{\partial p} p(1-p) = 0$$

$$\frac{\partial}{\partial p} p - \frac{\partial}{\partial p} p^2 = 0$$

$$1 - 2p = 0$$

$$\Rightarrow p = \frac{1}{2}$$