ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

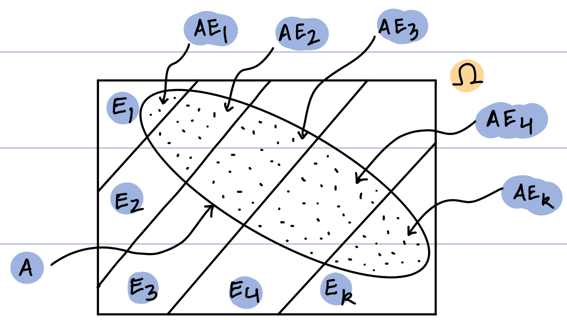
LECTURE 10: THE LAW OF TOTAL PROBABILITY, BAYES' THEOREM, AND INDEPENDENCE OF EVENTS

- TOPICS TO COVER (BASED ON CH 2.10)
  - → THE LAW OF TOTAL PROBABILITY
  - → BAYES' THEOREM
  - → INDEPENDENCE OF EVENTS

→ THE LAW OF TOTAL PROBABILITY

CONSIDER EVENTS $E_1, \dots, E_k$ SUCH THAT $E_i \cap E_j = \phi \quad \forall\, i, j = 1, \dots, k \quad i \neq j$

PARTITION OF Ω $\qquad \bigcup_{i=1}^{k} E_i = \Omega$



CONSIDER: $P(\Omega) = P\left(\bigcup_{i=1}^{k} E_i\right) = P(E_1) + P(E_2) + \cdots + P(E_k)$ AXIOM P.2

$\Rightarrow \quad P(E_1) + P(E_2) + \cdots + P(E_k) = 1$

NOTICE FROM THE ABOVE VENN DIAGRAM:

$A = AE_1 \cup AE_2 \cup \cdots \cup AE_k$, $AE_1, AE_2, \dots, AE_k$ ARE DISJOINT AS $E_1, E_2, \dots, E_k$ ARE

$\Rightarrow \quad P(A) = P(AE_1) + P(AE_2) + \cdots + P(AE_k)$ AXIOM P.2

$= P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2) + \cdots + P(A|E_k) \cdot P(E_k)$ DEF. OF COND. PROB.

$$P(A) = \sum_{i=1}^{k} P(A|E_i) \cdot P(E_i) \quad : \text{THE LAW OF TOTAL PROBABILITY} \quad \cdots (i)$$

TOTAL PROBABILITY FORMULA (TPF)

RECALL, IF $P(A) \neq 0$:

$$P(E_i|A) := \frac{P(AE_i)}{P(A)} \quad \because \text{DEF. OF COND. PROB.}$$

$$= \frac{P(A|E_i) \cdot P(E_i)}{P(A)} \quad \because \text{DEF. OF COND. PROB.}$$

$$P(E_i|A) = \frac{P(A|E_i) \cdot P(E_i)}{\sum_{i=1}^{k} P(A|E_i) \cdot P(E_i)} \quad \because \text{TPF} \quad : \text{BAYES' THEOREM}$$

EXAMPLE:

SEQUENTIAL EXPERIMENT

E_i : COIN i IS CHOSEN $\quad i = 1, 2 \quad P(E_1) = \frac{1}{5}, \quad P(E_2) = \frac{4}{5}$

A : HEAD $\quad P(A|E_1) = \frac{1}{3}, \quad P(A|E_2) = \frac{1}{4}$

$$P(A) = P(\text{HEAD}) \overset{\text{TPF}}{=} P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2)$$

$$= \underbrace{\frac{1}{3} \cdot \frac{1}{5}}_{(ii)} + \frac{1}{4} \cdot \frac{4}{5} \quad \cdots (iii)$$

$$P(E_1|A) = P(\text{YOU CHOSE COIN 1} \mid \text{EXP. RESULTED IN A HEAD})$$

$$= \frac{P(E_1 A)}{P(A)} \quad \because \text{DEF. OF COND. PROB.}$$

$$= \frac{P(A|E_1) \cdot P(E_1)}{P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2)} \quad \because \text{BAYES' RULE}$$

$$= \frac{(ii)}{(iii)}$$

INDEPENDENCE OF EVENTS

RE $\rightsquigarrow$ $(\Omega, \mathcal{F}, P)$, TWO EVENTS A and B.

1. $P(A) > 0$: $P(B|A) = \dfrac{P(AB)}{P(A)}$ DEF. OF COND. PROB.

• WE CALL B TO BE INDEP. OF A IF $\overbrace{\underbrace{P(B|A) = P(B)}_{\text{KNOWELEDGE OF A HAS NO EFFECT ON B}}}^{(iv)}$.

NOW, $P(B|A) = \dfrac{P(AB)}{P(A)} = P(B)$ UNDER (iv) : B IS INDEP. OF A

$\Rightarrow P(AB) = P(A) \cdot P(B)$ ... (I1)

• SIMILARLY, WE CALL A TO BE INDEP. OF B IF $\overbrace{\underbrace{P(A|B) = P(A)}_{\text{KNOWELEDGE OF B HAS NO EFFECT ON A}}}^{(v)}$.

ASSUMING $P(B) > 0$

NOW, $P(A|B) = \dfrac{P(AB)}{P(B)} = P(A)$ UNDER (v) : A IS INDEP. OF B

$\Rightarrow P(AB) = P(A) \cdot P(B)$ ... (I2)

D1

I1 and I2 are exactly the same, i.e., UNDER $P(A) > 0$ AND $P(B) > 0$:

IND. OF A FROM B (v) : $P(A|B) = P(A) \Rightarrow P(AB) = P(A) \cdot P(B)$ ... (I1)

(iv) IND. OF B FROM A : $P(B|A) = P(B) \Rightarrow P(AB) = P(A) \cdot P(B)$ ... (I2)

ALSO:

D2

IF $P(AB) = P(A) \cdot P(B)$ THEN $\underbrace{P(A|B) = P(A)}_{\substack{\text{IND. OF A FROM B} \\ (v)}}$ and $\underbrace{P(B|A) = P(B)}_{\substack{\text{IND. OF B FROM A} \\ (iv)}}$.

PROOF: EXERCISE!

IN SUMMARY:

$$\left.\begin{matrix} \underbrace{P(A|B) = P(A)}_{\substack{\text{IND. OF A FROM B} \\ (v)}} \\ \text{OR} \\ \underbrace{P(B|A) = P(B)}_{\substack{\text{IND. OF B FROM A} \\ (iv)}} \end{matrix}\right. \quad \underset{D2}{\overset{D1}{\rightleftarrows}} \quad P(AB) = P(A) \cdot P(B)$$

SO FAR

DEF. GIVEN $P(A) > 0$ AND $P(B) > 0$:

A and B ARE 'MUTUALLY' INDEPENDENT IF $P(AB) = P(A) \cdot P(B)$

2. $P(A) = 0$ $\qquad P(B|A) =$ UNDEFINED

CONSIDER A^c: $\underbrace{P(A^c) = 1 - P(A) = 1}$

DEF. OF COND. PROB.

$$P(B|A^c) = \frac{P(BA^c)}{P(A^c)} = \frac{P(BA^c)}{1} \quad \cdots (vi)$$

NOTE THAT $B \overset{\text{TPF}}{=} BA \cup BA^c \Rightarrow P(B) \overset{\text{AXIOM P.2}}{=} P(BA) + P(BA^c)$

$\therefore$ (vi) $\Rightarrow$ $P(B|A^c) = P(B) - P(AB)$

$= P(B) - P(A)\,P(B)$

IF WE USE THE DEF. OF INDEP. SO FAR

$\Rightarrow$ $P(B|A^c) = P(B)$ $\because P(A) = 0$

ALSO, $P(A^c|B) = P(A^c)$ (EXERCISE)

FINAL DEFINITION

IN GENERAL [$P(A) \geq 0$ AND $P(B) \geq 0$]

DEF. A and B ARE 'MUTUALLY' INDEPENDENT IF $P(AB) = P(A) \cdot P(B)$.

PROPERTY 1. IF A and B ARE INDEP. THEN A^c and B ARE ALSO INDEP.

PROOF. WE NEED TO SHOW : $P(A^cB) = P(A^c) \cdot P(B)$

CONSIDER: $B = AB \cup A^cB$

$P(B) = P(AB) + P(A^cB)$ $\because$ AXIOM P.2

$\Rightarrow$ $P(A^cB) = P(B) - P(AB)$

$= P(B) - P(A) \cdot P(B)$ $\because$ INDEP. OF A and B

$= (1 - P(A)) \cdot P(B)$

$\Rightarrow$ $P(A^cB) = P(A^c)\,P(B)$

HENCE THE PROPERTY.

PROPERTY 2 : IF A and B ARE INDEP. $\Rightarrow$ A and B^c ARE INDEP.

PROPERTY 3 : IF A and B ARE INDEP. $\Rightarrow$ A^c and B^c ARE INDEP.

DEFINITION: $\underbrace{A, B, \text{ AND } C}_{\text{3 EVENTS}}$ ARE PAIRWISE INDEP. IF

$\Rightarrow \binom{3}{2}$ PAIRS $= 3$

$$P(AB) = P(A) \cdot P(B)$$

$$P(BC) = P(B) \cdot P(C)$$

$$P(AC) = P(A) \cdot P(C)$$

STRONGER THAN PAIRWISE INDEP.

DEFINITION: A, B, AND C ARE INDEP. IF

$$P(AB) = P(A) \cdot P(B)$$

$$P(BC) = P(B) \cdot P(C)$$

$$P(AC) = P(A) \cdot P(C)$$

$$P(ABC) = P(A) \cdot P(B) \cdot P(C)$$

OBSERVE THAT : INDEP. $\Rightarrow$ PAIRWISE INDEP.

PAIRWISE INDEP. $\not\Rightarrow$ INDEP.

PROPERTY 4 : A, B, AND C ARE INDEP. (*) THEN A IS INDEP. OF $B \cdot C$.

PROOF :

$$P(ABC) = P(A) \cdot \underbrace{P(B) \cdot P(C)}_{(*)}$$

$$\Rightarrow P(ABC) = P(A) \cdot P(BC)$$

PROPERTY 5 : $\underbrace{A, B, \text{ AND } C \text{ ARE INDEP.}}_{(*)}$ THEN A AND $B \cup C$ ARE ALSO INDEP.

PROOF : WE NEED TO SHOW: $P(A(B \cup C)) = P(A) \cdot P(B \cup C)$

CONSIDER :

$$P(A \cdot (B \cup C)) = P(AB \cup AC)$$

ADDITION THEOREM

$$= P(AB) + P(AC) - P(\underbrace{AB \cdot AC}_{ABC})$$

(*) (*) (*)

$$= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$$

$$= P(A) \left[P(B) + P(C) - P(BC) \right]$$

ADDITION THEOREM

$$\Rightarrow P(A \cdot (B \cup C)) = P(A) \; P(B \cup C)$$

FINAL OBSERVATION: IF A, B, AND C ARE INDEP. THEN A (or A^c) IS INDEP. OF ANY EVENT FORMED USING SET OPERATIONS ON B and C.

STRONGER THAN PAIRWISE INDEP.

DEFINITION. $A_1, A_2, \cdots, A_n$ are INDEP. IF

$$P(A_{i_1} A_{i_2} \cdots A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \cdots \cdot P(A_{i_k})$$

$$2 \leq k \leq n \quad \text{and} \quad 1 \leq i_1 < i_2 < \dots < i_k \leq n$$

THERE ARE $\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$ CONDITIONS

WEAKER THAN INDEP.

DEFINITION. $A_1, A_2, \cdots, A_n$ are PAIRWISE INDEP. IF

$$P(A_{i_1} A_{i_2}) = P(A_{i_1}) \cdot P(A_{i_2})$$

$$1 \leq i_1 < i_2 \leq n$$

THERE ARE $\binom{n}{2}$ CONDITIONS

INDEPENDENCE OF RANDOM VARIABLES

DEFINITION. RANDOM VARIABLES X and Y ARE INDEP. IF ANY EVENT OF THE FORM $X \in A$ IS INEP. OF ANY EVENT OF THE FORM $X \in B$.

$$X \in A \quad : \quad \{\omega : X(\omega) \in A\} \in \mathcal{F}$$

$$Y \in B \quad : \quad \{\omega : Y(\omega) \in B\} \in \mathcal{F}$$

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \qquad \forall\ A, B \subseteq \mathbb{R}$$

PROPERTY: IF X and Y ARE INDEP. THEN

$$\underbrace{P\{X = i, Y = j\}}_{\text{JOINT PMF}:\ p_{XY}(i,j)} = \underbrace{P\{X = i\}}_{p_X(i)} \cdot \underbrace{P\{Y = j\}}_{p_Y(j)} \qquad i, j \in \mathbb{R}$$

PMFS OF X and Y

PROOF: GIVEN THAT X AND Y ARE INDEP.

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \qquad \forall\ A, B \subseteq \mathbb{R}$$

TAKE $A = \{i\}$, $B = \{j\}$ $\quad i, j \in \mathbb{R}$

$$P\{X = i, Y = j\} = P\{X = i\} \cdot P\{Y = j\}$$

PROPERTY: IF $\underbrace{P\{X = i, Y = j\}}_{\text{JOINT PMF}:\ p_{XY}(i,j)} = \underbrace{P\{X = i\}}_{p_X(i)} \cdot \underbrace{P\{Y = j\}}_{p_Y(j)} \quad i, j \in \mathbb{R}$

PMFS OF X and Y

THEN X and Y ARE INDEP.

PROOF : WE NEED TO SHOW:

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \quad \forall \; A, B \subseteq \mathbb{R}$$

AXIOM P.2

CONSIDER : $P\{X \in A, Y \in B\} = \sum_{(i,j):\, i \in A,\, j \in B} P\{X = i, Y = j\}$

GIVEN

$$= \sum_{(i,j):\, i \in A,\, j \in B} P\{X = i\} P\{Y = j\}$$

$$= \sum_{i:\, i \in A} P\{X = i\} \cdot \sum_{j:\, j \in B} P\{Y = j\}$$

$$\Rightarrow P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{X \in B\}$$