University of Illinois **Fall 2025**

ECE 313: Problem Set 13

Due: Monday, December 8 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.9.2, 4.9.3, 4.10.1, 4.10.2, and 4.11

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**Covariance and Correlation**]
   Random variables X_1 and X_2 represent two observations of a signal corrupted by noise. They have the same mean μ and variance σ^2. The signal-to-noise ratio (SNR) of the observation X_1 or X_2 is defined as the ratio $\mathrm{SNR}_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1+X_2}{2}$.

   (a) Show that the SNR of S is twice that of the individual observations, if X_1 and X_2 are uncorrelated.

   (b) The system designer notices that the averaging strategy is giving $\mathrm{SNR}_S = (1.5)\mathrm{SNR}_X$. She correctly assumes that the observations X_1 and X_2 are correlated. Determine the value of the correlation coefficient ρ_{X_1,X_2}.

   (c) Under what condition on ρ_{X_1,X_2} can the averaging strategy result in an SNR_S that is as high as possible?

2. [**MMSE Estimation**]
   Consider the joint pdf below describing the dependence between random variables X and Y:

$$f_{X,Y} = \begin{cases} u+v, & (u,v) \in [0,1]^2; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   We wish to design various types of minimum mean squared error (MMSE) estimators of Y.

   (a) Determine the MMSE optimal constant estimator c^* of Y and the resulting MSE.

   (b) Determine the MMSE optimal unconstrained estimator $g^*(X)$ and the resulting MSE. Compare the MSE values obtained in parts (a) and (b). *Hint: you can use a numerical solver to evaluate the integral that appears in the MSE.*

(c) Determine the MMSE linear estimator $L^*(X)$ and the resulting MSE.

3. **[MMSE Estimation]**
Let $Y \sim \text{Exp}(\lambda = 2)$ and $Z \sim \text{Exp}(\lambda = 1)$ be independent random variables. Let the observation be $X = Y + Z$.

   (a) Determine the MMSE optimal constant estimator c^* of Y and the resulting MSE.

   (b) Determine the MMSE-optimal (unconstrained) estimator $g^*(X)$. *(Note: You do not need to compute the resulting MSE; it is computable but involves a more complicated integral.)*

   (c) Determine the MMSE linear estimator $L^*(X)$ and the resulting MSE.

4. **[Gaussian random variables]**
Let X and Y be independent random variables with $X, Y \sim \mathcal{N}(0, 1)$.

   (a) Find $\text{Cov}(3X - Y,\ X + 4Y + 2)$.

   (b) Express $\mathbb{P}(3X + Y \geq 2)$ in terms of the Q-function.

   (c) Express $\mathbb{P}\big((2X + Y)^2 > 9\big)$ in terms of the Q-function.

5. **[Gaussian Random variables and Estimation]**
Suppose that X and Y are jointly Gaussian random variables with

$$\mathbb{E}[X] = 1, \quad \mathbb{E}[Y] = -2, \quad \text{Var}(X) = 4, \quad \text{Var}(Y) = 9,$$

and correlation coefficient $\rho = \frac{1}{3}$. Let $W = 3X + Y - 2$.

   (a) Find $\mathbb{E}[W]$ and $\text{Var}(W)$.

   (b) Find the best unconstrained estimator $g^\star(W)$ of Y based on W and the resulting MSE.