University of Illinois **Fall 2025**

ECE 313: Problem Set 12: Problems and Solutions

Due: Monday, Dec 1 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.6 – 4.7.1

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday (except this one). You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[More Problems on Joint Densities]**
Rayleigh fading and selection diversity: The signal strength received at a base station from a cellular phone in a dense urban environment can be modeled quite well by a *Rayleigh* random variable. When a cellular phone user is around the midpoint between two base stations, some systems use *selection diversity*, where the base station with the larger received signal strength is used to decode the user's message. Let X and Y denote the signal strengths received at the two base stations. Then

$$f_X(u) = f_Y(u) = \begin{cases} ue^{-\frac{u^2}{2}}, & \text{if } u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that X and Y are independent.

(a) Find the mean signal strength at each base-station, i.e., find $E[X]$.
Hint: Convert the integral required to compute $E[X]$ into one that corresponds to finding the variance of a $N(0,1)$ random variable.
Solution: Using the hint

$$\begin{aligned} E[X] &= \int_0^\infty x^2\, e^{-\frac{x^2}{2}}\, dx = \frac{1}{2}\int_{-\infty}^\infty x^2\, e^{-\frac{x^2}{2}}\, dx \\ &= \frac{\sqrt{2\pi}}{2}\int_{-\infty}^\infty x^2\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\, dx \\ &= \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}} \approx 1.253. \end{aligned}$$

(b) Find the pdf of the signal strength chosen by the selection diversity system, i.e., find the pdf of $Z = \max(X, Y)$.

Solution: We first compute the CDF of Z as

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{\max(X,Y) \le z\} \\ &= P(X \le z, Y \le z) = F_X(z)\ F_Y(z) \\ &= \left(1 - e^{-\frac{z^2}{2}}\right)^2 \quad \text{(apply substitution } v = -\frac{u^2}{2} \text{ for computing each CDF from pdf)} \end{aligned}$$

for $z \ge 0$, and $F_Z(z) = 0$ for $z < 0$. Differentiating we get

$$f_Z(z) = 2\ \left(1 - e^{-\frac{z^2}{2}}\right)\ e^{-\frac{z^2}{2}}\ z = 2ze^{-\frac{z^2}{2}} - 2ze^{-z^2}$$

for $z \ge 0$, and $f_Z(z) = 0$ for $z < 0$. It is easy to check that f_Z integrates to 1.

(c) Find the mean signal strength after selection diversity, i.e., find $E[Z]$. Again, you may want to use the hint given in part (a).

Solution: Using the same technique as in part (a)

$$\begin{aligned} E[Z] &= \int_0^\infty 2\ z^2\ e^{-\frac{z^2}{2}} dz - \int_0^\infty 2\ z^2\ e^{-z^2} dz \\ &= \int_{-\infty}^\infty z^2\ e^{-\frac{z^2}{2}} dz - \int_{-\infty}^\infty z^2\ e^{-z^2} dz \\ &= \sqrt{2\pi} \int_{-\infty}^\infty z^2\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{z^2}{2}} dz - \sqrt{\pi} \int_{-\infty}^\infty z^2\ \frac{1}{\sqrt{\pi}}\ e^{-z^2} dz \\ &= \sqrt{2\pi} - \frac{\sqrt{\pi}}{2} \approx 1.62. \end{aligned}$$

We see that $E[Z] > E[X]$ as expected.

2. **[More Problems on Joint Densities]**

Suppose X and Y are jointly continuous random variables with joint pdf $f_{X,Y}(u,v)$, and assume that the support of (X,Y) is contained in $(0,\infty) \times (0,\infty)$. Define

$$Z = \frac{X}{Y}.$$

(a) Express $f_Z(c)$ in terms of $f_{X,Y}(u,v)$

Solution: We can find the CDF of Z first.

$$\begin{aligned} F_Z(c) &= P\{Z \le c\} = P\{\frac{X}{Y} \le c\} = P\{X \le cY\} \\ &= \int_0^\infty \int_0^{cv} f_{X,Y}(u,v) du dv \end{aligned}$$

The target pdf is the derivative of $F_Z(c)$

$$\begin{aligned} f_Z(c) &= \frac{dF_Z(c)}{dc} = \\ &= \int_0^\infty \frac{d}{dc} \int_0^{cv} f_{X,Y}(u,v) du dv \\ &= \int_0^\infty v f_{X,Y}(cv,v) dv \end{aligned}$$

Geometrically, this integrates the joint pdf along the ray $u = cv$ in the (u,v)-plane, with a weighting factor v.

(b) Consider

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{\pi} & \text{if } \sqrt{(u-1)^2+(v-2)^2} \le 1 \\ 0 & \text{else} \end{cases}$$

In other words, (X,Y) is uniformly distributed over the disk of radius 1 centered at $(1,2)$. Find $f_Z(0.5)$.

Solution: From part (a), for $c > 0$,

$$f_Z(c) = \int_0^\infty v\, f_{X,Y}(cv, v)\, dv.$$

For $c = 0.5$, this becomes

$$f_Z(0.5) = \int_0^\infty v\, f_{X,Y}(0.5v, v)\, dv.$$

The integrand is nonzero only when $(u,v) = (0.5v, v)$ lies inside the disk

$$(u-1)^2 + (v-2)^2 \le 1,$$

i.e.,

$$(0.5v-1)^2 + (v-2)^2 \le 1.$$

Expanding and simplifying:

$$5v^2 - 20v + 16 = 0.$$

$$v = 2 \pm \frac{2\sqrt{5}}{5} = 2 \pm \frac{2}{\sqrt{5}}.$$

$$v_L = 2 - \frac{2}{\sqrt{5}}, \quad v_R = 2 + \frac{2}{\sqrt{5}}.$$

On this segment, $f_{X,Y}(0.5v, v) = \frac{1}{\pi}$, so

$$\begin{aligned} f_Z(0.5) &= \int_{v_L}^{v_R} v \cdot \frac{1}{\pi}\, dv = \frac{1}{\pi} \int_{v_L}^{v_R} v\, dv \\ &= \frac{1}{\pi} \left[\frac{v^2}{2} \right]_{v_L}^{v_R} = \frac{1}{2\pi} \left(v_R^2 - v_L^2 \right) \\ &= \frac{8\sqrt{5}}{5\pi}. \end{aligned}$$

3. **[Joint PDFs of (linear) Functions of Random Variables]**
Consider the joint pdf of X and Y given below:

$$f_{X,Y} = \begin{cases} 2e^{-(u+v)}, & 0 < u < v < \infty \\ 0, & \text{else} \end{cases}$$

(a) Are X and Y independent? Give reasons.

Solution: X and Y are dependent since the support $\mathcal{S}_{xy}$ of its joint pdf is not a product set. To see this, choose two points $(0.1, 0.2) \in \mathcal{S}_{xy}$ and $(0.4, 0.5) \in \mathcal{S}_{xy}$. Applying the swap test results in the new points $(0.1, 0.5) \in \mathcal{S}_{xy}$ and $(0.4, 0.2) \notin \mathcal{S}_{xy}$. Hence, the swap test fails.

(b) Find the joint pdf of $W = 2X + Y$ and $Z = X - Y$. Are W and Z independent?

Solution: One can show that,

$$f_{W,Z}(\alpha, \beta) = \frac{1}{3} f_{X,Y}\left(\frac{\alpha+\beta}{3}, \frac{\alpha-2\beta}{3}\right) = \begin{cases} \frac{2}{3} e^{-(\frac{2\alpha-\beta}{3})}, & \beta < 0, \alpha > -\beta \\ 0, & \text{else} \end{cases}$$

W and Z are dependent since the support $\mathcal{S}_{wz}$ of its joint pdf is not a product set. Applying the swap test to $(0.3, -0.2) \in \mathcal{S}_{wz}$ and $(0.5, -0.4) \in \mathcal{S}_{wz}$, we obtain $(0.3, -0.4) \notin \mathcal{S}_{wz}$ and $(0.5, -0.2) \in \mathcal{S}_{wz}$ which fails the test.

4. **[Joint PDFs of functions of random variables]**
Suppose X and Y have joint pdf $f_{X,Y}$, and $W = 3X - 2Y$ and $Z = 4X + Y$.

(a) Find the pdf of Z. Express it in terms of $f_{X,Y}$.

Solution: We first find the CDF of Z and then differentiate it to get the pdf. For any $\alpha \in \mathbb{R}$, the event $\{Z \le \alpha\}$ is the same as the event that the random point (X, Y) in the plane falls into the region where $4x + y \le \alpha$.

For each u fixed, integrate over v from $-\infty$ to $\alpha - 4u$:

$$F_Z(\alpha) = \mathbb{P}\{Z \le \alpha\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\alpha-4u} f_{X,Y}(u, v)\, dv \right) du.$$

Therefore,

$$\begin{aligned} f_Z(\alpha) = \frac{dF_Z(\alpha)}{d\alpha} &= \int_{-\infty}^{\infty} \frac{d}{d\alpha} \left(\int_{-\infty}^{\alpha-4u} f_{X,Y}(u, v)\, dv \right) du \\ &= \int_{-\infty}^{\infty} f_{X,Y}(u, \alpha - 4u)\, du. \end{aligned}$$

Alternatively, for each v fixed, integrate over u from $-\infty$ to $(\alpha - v)/4$:

$$F_Z(\alpha) = \mathbb{P}\{Z \le \alpha\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{(\alpha-v)/4} f_{X,Y}(u, v)\, du \right) dv.$$

Therefore,

$$\begin{aligned} f_Z(\alpha) = \frac{dF_Z(\alpha)}{d\alpha} &= \int_{-\infty}^{\infty} \frac{d}{d\alpha} \left(\int_{-\infty}^{(\alpha-v)/4} f_{X,Y}(u, v)\, du \right) dv \\ &= \frac{1}{4} \int_{-\infty}^{\infty} f_{X,Y}\left(\frac{\alpha - v}{4}, v \right) dv. \end{aligned}$$

(b) Express the joint PDF of W and Z in terms of $f_{X,Y}$.

Solution: Write

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}.$$

We apply the linear change of variables formula

$$f_{W,Z}(\alpha, \beta) = \frac{1}{|\det A|} f_{X,Y}\big(A^{-1}(\alpha, \beta)^{\mathsf{T}}\big).$$

First compute

$$\det(A) = 3 \cdot 1 - (-2) \cdot 4 = 11, \qquad A^{-1} = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}.$$

Hence,

$$A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{11} \begin{pmatrix} \alpha + 2\beta \\ -4\alpha + 3\beta \end{pmatrix}.$$

Specifically, the random variables W and Z have joint pdf given by

$$f_{W,Z}(\alpha, \beta) = \frac{1}{11} f_{X,Y}\left(\frac{\alpha + 2\beta}{11}, \frac{-4\alpha + 3\beta}{11}\right).$$