ECE 313: Problem Set 12

Due: Monday, Dec 1 at 7:00:00 p.m.

Reading: *ECE 313 Course Notes,* Sections 4.6 – 4.7.1

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday (except this one). You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:

NAME

NETID

SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**More Problems on Joint Densities**]

   Rayleigh fading and selection diversity: The signal strength received at a base station from a cellular phone in a dense urban environment can be modeled quite well by a *Rayleigh* random variable. When a cellular phone user is around the midpoint between two base stations, some systems use *selection diversity*, where the base station with the larger received signal strength is used to decode the user's message. Let X and Y denote the signal strengths received at the two base stations. Then

   $$f_X(u) = f_Y(u) = \begin{cases} ue^{-\frac{u^2}{2}}, & \text{if } u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   Assume that X and Y are independent.

   (a) Find the mean signal strength at each base-station, i.e., find $E[X]$.
   Hint: Convert the integral required to compute $E[X]$ into one that corresponds to finding the variance of a $N(0,1)$ random variable.

   (b) Find the pdf of the signal strength chosen by the selection diversity system, i.e., find the pdf of $Z = \max(X, Y)$.

   (c) Find the mean signal strength after selection diversity, i.e., find $E[Z]$. Again, you may want to use the hint given in part (a).

2. [**More Problems on Joint Densities**]

   Suppose X and Y are jointly continuous random variables with joint pdf $f_{X,Y}(u, v)$, and assume that the support of (X, Y) is contained in $(0, \infty) \times (0, \infty)$. Define

   $$Z = \frac{X}{Y}.$$

(a) Express $f_Z(c)$ in terms of $f_{X,Y}(u,v)$

(b) Consider

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{\pi} & \text{if } \sqrt{(u-1)^2 + (v-2)^2} \leq 1 \\ 0 & \text{else} \end{cases}$$

In other words, (X, Y) is uniformly distributed over the disk of radius 1 centered at $(1, 2)$. Find $f_Z(0.5)$.

3. **[Joint PDFs of (linear) Functions of Random Variables]**
   Consider the joint pdf of X and Y given below:

   $$f_{X,Y} = \begin{cases} 2e^{-(u+v)}, & 0 < u < v < \infty \\ 0, & \text{else} \end{cases}$$

   (a) Are X and Y independent? Give reasons.

   (b) Find the joint pdf of $W = 2X + Y$ and $Z = X - Y$. Are W and Z independent?

4. **[Joint PDFs of functions of random variables]**
   Suppose X and Y have joint pdf $f_{X,Y}$, and $W = 3X - 2Y$ and $Z = 4X + Y$.

   (a) Find the pdf of Z. Express it in terms of $f_{X,Y}$.

   (b) Express the joint PDF of W and Z in terms of $f_{X,Y}$.