University of Illinois **Fall 2025**

ECE 313: Problem Set 11: Problems and Solutions

Due: Friday, November 21 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.4 – 4.6

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[Joint PDF and Independence]**
Let (X, Y) be continuous random variables with joint probability density function (PDF) given by:

$$f_{X,Y}(x, y) = \begin{cases} 6(1-y), & 0 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Answer the following questions about X and Y.

(a) Verify that $f_{X,Y}(x, y)$ is a valid joint PDF.
Solution: We verify that the total integral over the support equals 1:

$$\int_0^1 \int_0^y 6(1-y)\,dx\,dy = \int_0^1 6(1-y)\cdot y\,dy = 6\int_0^1 y(1-y)\,dy$$

$$= 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 6\left(\frac{1}{2} - \frac{1}{3}\right) = 6 \cdot \frac{1}{6} = 1$$

So $f_{X,Y}(x, y)$ is a valid joint PDF.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
Solution: Marginal of X:

$$f_X(x) = \int_x^1 6(1-y)\,dy = 6\left[y - \frac{y^2}{2}\right]_x^1 = 6\left(\frac{1}{2} - x + \frac{x^2}{2}\right)$$

for $0 \le x \le 1$.

Marginal of Y:

$$f_Y(y) = \int_0^y 6(1-y)\,dx = 6(1-y)\cdot y = 6y(1-y)$$

for $0 \le y \le 1$.

(c) Find the conditional PDF $f_{X|Y}(x|y)$.

Solution:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{6(1-y)}{6y(1-y)} = \frac{1}{y}, \quad \text{for } 0 \le x \le y$$

So:

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 \le x \le y \\ 0, & \text{otherwise} \end{cases}$$

This is the PDF of a uniform distribution on $[0, y]$, so $X|Y = y \sim \text{Uniform}(0, y)$.

(d) Find $\mathbb{E}[X|Y = y]$.

Solution: Since $X|Y = y \sim \text{Uniform}(0, y)$, the conditional expectation is:

$$\mathbb{E}[X|Y = y] = \frac{0 + y}{2} = \frac{y}{2}$$

(e) Are X and Y independent? Justify your answer.

Solution: No, X and Y are not independent. The joint PDF does not factor into a product of functions of x and y. Also, the support is triangular: $0 \le x \le y \le 1$, which shows that the values of X are constrained by Y, indicating dependence.

2. **[Joint PDF and Independence]**

Let (X, Y) be continuous random variables with joint probability density function (PDF) given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{xy}{c}, & \text{if } x \in [0,1] \text{ and } y \in [0,2] \\ \frac{3y-xy}{c} & \text{if } x \in [2,3] \text{ and } y \in [0,2] \\ 0, & \text{otherwise} \end{cases}$$

Answer the following questions about X and Y.

(a) Compute c.

Solution: The total integral over the support equals to 1:

$$\begin{aligned} \int_0^1 \int_0^2 \frac{xy}{c} dydx + \int_2^3 \int_0^2 \frac{3y - xy}{c} dydx &= \int_0^1 \frac{x}{c} \frac{y^2}{2}\Big|_{y=0}^2 dx + \int_2^3 \frac{3-x}{c} \frac{y^2}{2}\Big|_{y=0}^2 dx \\ &= \int_0^1 \frac{2x}{c} dx + \int_2^3 2 \times \frac{3-x}{c} dx \\ &= \frac{x^2}{c}\Big|_{x=0}^1 - \frac{(3-x)^2}{c}\Big|_{x=2}^3 \\ &= \frac{1}{c} + \frac{1}{c} = 1 \\ c &= 2 \end{aligned}$$

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution: **Marginal of X:**

Since $f_{XY}(x, y)$ follows different equations in different x ranges, the marginal of X will need to be computed separately. For $x \in [0, 1]$

$$
\begin{aligned}
f_X(x) &= \int_0^2 f_{XY}(x, y)dy = \int_0^2 \frac{xy}{2} dy \\
&= x\frac{y^2}{4}\bigg|_{y=0}^2 = x
\end{aligned}
$$

For $x \in [2, 3]$

$$
\begin{aligned}
f_X(x) &= \int_0^2 f_{XY}(x, y)dy = \int_0^2 \frac{3y - xy}{2} dy \\
&= (3 - x)\frac{y^2}{4}\bigg|_{y=0}^2 = 3 - x
\end{aligned}
$$

Combine together we get the f_X

$$
f_X(x) = \begin{cases} x, & \text{if } x \in [0, 1] \\ 3 - x & \text{if } x \in [2, 3] \\ 0, & \text{otherwise} \end{cases}
$$

Marginal of Y: For $y \in [0, 2]$

$$
\begin{aligned}
f_Y(y) &= \int_0^1 \frac{xy}{2} dx + \int_2^3 \frac{(3 - x)y}{2} dx \\
&= y\frac{x^2}{4}\bigg|_{x=0}^1 - y\frac{(3 - x)^2}{4}\bigg|_{x=2}^3 \\
&= \frac{y}{4} + \frac{y}{4} = \frac{y}{2}
\end{aligned}
$$

The formal $f_Y(y)$ is

$$
f_Y(y) = \begin{cases} \frac{y}{2}, & \text{if } y \in [0, 2] \\ 0, & \text{otherwise} \end{cases}
$$

(c) Find the conditional PDF $f_{X|Y}(x)$ and decide if X and Y are independent.

Solution: For $y \in [0, 2]$,

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\
&= \begin{cases} \frac{xy/2}{y/2}, & \text{if } x \in [0, 1] \\ \frac{(3-x)y/2}{y/2} & \text{if } x \in [2, 3] \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

The formal conditional PDF is

$$
f_{X|Y}(x|y) = \begin{cases} x, & \text{if } x \in [0, 1] \text{ and } y \in [0, 2] \\ 3 - x & \text{if } x \in [2, 3] \text{ and } y \in [0, 2] \\ 0, & \text{otherwise} \end{cases}
$$

This conditional distribution is exactly the same as f_X, so X and Y are independent.

3. **[Sum of Two Independent Continuous-type Random Variables]**
   Suppose X and Y have the joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 4uv, & \text{if } 0 \le u \le 1,\ 0 \le v \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

   Let $S = X + Y$. Find the pdf of S, i.e., find $f_S(s)$.

   (a) Find the pdf of S, i.e., find $f_S(s)$. *Hint:* You may want to consider the cases $s < 0$, $0 \le s < 1$, $1 \le s \le 2$, and $s > 2$ separately.
   Solution: Clearly $f_S(s) = 0$ for $s < 0$ and $s > 2$.
   For $0 \le s \le 2$,

$$f_S(s) = \int_{u=-\infty}^{\infty} f_{X,Y}(u, s-u)du.$$

   Therefore, for $0 \le s < 1$,

$$f_S(s) = \int_{u=0}^{s} 4u(s-u)du = 4\left[\frac{su^2}{2} - \frac{u^3}{3}\right]_0^s = 4\left[\frac{s^3}{2} - \frac{s^3}{3}\right] = \frac{2}{3}s^3.$$

   For $1 \le s \le 2$,

$$f_S(s) = \int_{u=s-1}^{1} 4u(s-u)du = 4\left[\frac{su^2}{2} - \frac{u^3}{3}\right]_{u=s-1}^{1} = -\frac{2}{3}s^3 + 4s - \frac{8}{3}.$$

   Putting it all together, we have:

$$f_S(s) = \begin{cases} \frac{2}{3}s^3, & \text{if } 0 \le s < 1; \\ -\frac{2}{3}s^3 + 4s - \frac{8}{3}, & \text{if } 1 \le s \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Confirm that $f_S(s)$ integrates to 1 and therefore is a valid pdf.
   Solution: We compute:

$$\int_{-\infty}^{\infty} f_S(s)\,ds = \int_0^1 \frac{2}{3}s^3\,ds + \int_1^2 \left(-\frac{2}{3}s^3 + 4s - \frac{8}{3}\right) ds.$$

   Integration over $[0,1]$ yields:

$$\int_0^1 \frac{2}{3}s^3\,ds = \frac{2}{3}\left[\frac{s^4}{4}\right]_0^1 = \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}.$$

   Integration over $[1,2]$ yields:

$$\int_1^2 \left(-\frac{2}{3}s^3 + 4s - \frac{8}{3}\right) ds = \left[-\frac{1}{6}s^4 + 2s^2 - \frac{8}{3}s\right]_1^2.$$

Evaluating at the bounds:

$$\begin{aligned}
\text{At } s = 2: &\quad -\frac{1}{6}(16) + 2(4) - \frac{8}{3}(2) = -\frac{16}{6} + 8 - \frac{16}{3} = 0, \\
\text{At } s = 1: &\quad -\frac{1}{6}(1) + 2(1) - \frac{8}{3}(1) = -\frac{1}{6} + 2 - \frac{8}{3} = -\frac{5}{6}.
\end{aligned}$$

Thus,

$$\int_1^2 \left(-\frac{2}{3}s^3 + 4s - \frac{8}{3}\right) ds = 0 - \left(-\frac{5}{6}\right) = \frac{5}{6}.$$

$$\int_{-\infty}^{\infty} f_S(s)\, ds = \frac{1}{6} + \frac{5}{6} = 1.$$

Since $f_S(s) \geq 0$ for all s and

$$\int_{-\infty}^{\infty} f_S(s)\, ds = 1,$$

we therefore conclude that $f_S(s)$ is a valid pdf.

4. **[Sum of Two Continuous-type Random Variables]**
Suppose X and Y have the joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 2v, & 0 \leq u \leq 1,\ 0 \leq v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $S = X + Y$.

(a) Are X and Y independent? Justify your answer.
Solution: We compare $f_{X,Y}(u,v)$ to $f_X(u)f_Y(v)$. For $0 \leq u \leq 1$,

$$f_X(u) = \int_0^1 2v\, dv = 1,$$

and for $0 \leq v \leq 1$,

$$f_Y(v) = \int_0^1 2v\, du = 2v.$$

Thus

$$f_X(u)f_Y(v) = 2v = f_{X,Y}(u,v).$$

Hence, X and Y are independent.

(b) Find the pdf of S, i.e., compute $f_S(s)$.
Solution: Clearly $f_S(s) = 0$ for $s < 0$ or $s > 2$. For $0 \leq s \leq 2$,

$$f_S(s) = \int_{-\infty}^{\infty} f_{X,Y}(u, s-u)\, du.$$

For $0 \leq s < 1$, the valid region is $0 \leq u \leq s$, hence

$$f_S(s) = \int_0^s 2(s-u)\, du = \int_0^s 2t\, dt = s^2.$$

For $1 \leq s \leq 2$, the valid region is $s-1 \leq u \leq 1$, hence

$$f_S(s) = \int_{s-1}^{1} 2(s-u)\,du = \int_{s-1}^{1} 2t\,dt = 1-(s-1)^2.$$

Putting everything together,

$$f_S(s) = \begin{cases} s^2, & 0 \leq s < 1, \\ 1-(s-1)^2, & 1 \leq s \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$