University of Illinois **Fall 2025**

ECE 313: Problem Set 10: Problems and Solutions

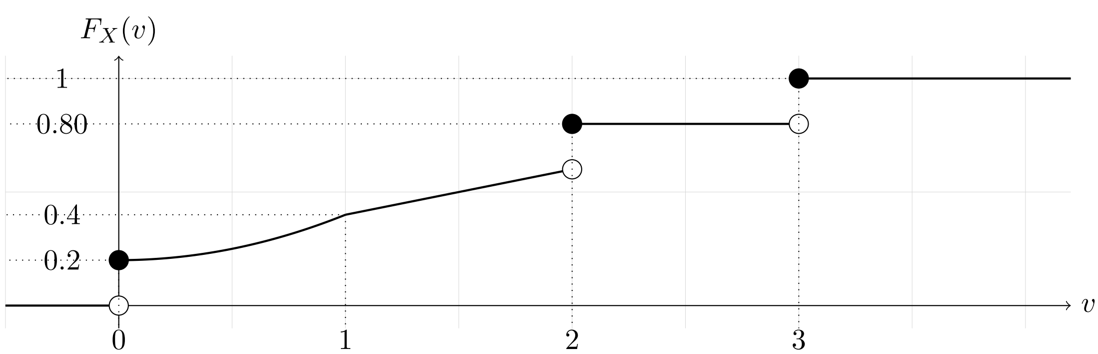
Due: Friday, Nov 14 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.8.2 - 4.1

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[Generating samples of a RV]**
   X is a random variable with CDF shown as below:

$$F_X(v) = \begin{cases} 0, & v < 0, \\ 0.2 + 0.2v^2, & 0 \leq v \leq 1, \\ 0.4 + 0.2(v-1), & 1 < v < 2, \\ 0.80, & 2 \leq v < 3, \\ 1, & v \geq 3. \end{cases}$$



   (a) For the CDF $F_X(v)$ of X shown above, find the function $g(U)$, where $U \sim \text{Unif}[0,1]$, such that $v = g(u)$, i.e., samples v of X are generated when samples u of U are passed through the function $g(.)$.

   Solution: For $u \in (0,1)$, a suitable function $g(u)$ can be found such that $X = g(U)$

has the CDF shown in the figure. Specifically,

$$g(u) = F_X^{-1}(u) = \min\{v : F_X(v) \geq u\}$$

Therefore, using the provided CDF, we can obtain $g(u)$ as follows:

$$g(u) = \begin{cases} 0, & 0 < u \leq 0.2 \\ \sqrt{5u-1}, & 0.2 < u < 0.4 \\ 5u-1, & 0.4 \leq u < 0.6 \\ 2, & 0.6 \leq u \leq 0.8 \\ 3, & 0.8 < u \leq 1 \end{cases}$$

(b) Prove that the function $g()$ obtained in part (a) does in fact generate samples of X from samples of U. *Hint*: show that $X = g(U)$ has the CDF shown in the figure above.

Solution: One can prove that $g(.)$ generates samples of X by considering each region of $g(u)$ separately, and showing that $F_X(v)$ takes the correct form defined in the problem statement. For example, consider $u \in [0.4, 0.6)$. Then,

$$F_X(v) = P\{X \leq v\} = P\{g(U) \leq v\} = P\{5U - 1 \leq v\} = P\{U \leq \frac{v+1}{5}\}$$
$$= \frac{v+1}{5} = 0.2v + 0.2$$

which is the expression for $F_X(v)$ for $v = g(u) \in [1, 2)$. Similarly, for $u \in [0.8, 1)$, we have $g(u) = 3$ and

$$F_X(v) = P\{X \leq 3\} = 1$$

which is the expression for $F_X(v = 3)$.

2. **[Joint CDF and PMF]**
Roll a fair six-sided die twice. Let X and Y denote the outcome of the 1st and 2nd die roll, respectively. Define $Z = 2X - Y$ as the sum of two rolls. Answer the following questions about X and Z.

(a) Decide the support in the (u, v) plane of the joint pmf $p_{X,Z}(u, v)$.

Solution: The support is all pairs of integers satisfying $1 \leq u \leq 6$ and $2u-6 \leq v \leq 2u-1$

(b) Find the joint pmf $p_{X,Z}(u, v)$.

Solution: Since $X = u$ and $Z = v$ defines the corresponding Y value $Y = 2X - Z$, the joint pmf is flat within the support.

$$p_{X,Z}(u, v) = \begin{cases} \frac{1}{36} & 1 \leq u \leq 6 \text{ and } 2u - 6 \leq v \leq 2u - 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the pmf $p_Z(v)$.

Solution: The support of Z is $[1 \times 2 - 6, 6 \times 2 - 1] = [-4, 11]$. For $v \in [0, 7]$, there will be exactly three (u, v) pairs such that the second roll $u - v \in [1, 6]$ is valid. Accordingly, $p_Z(v|v \in [0, 7]) = \frac{1}{36} \times 3 = \frac{1}{12}$

For marginal cases where $v \in \{-4, -3, 10, 11\}$, the only valid $X = u$ are $1, 1, 6, 6$ respectively. When $v \in \{-2, -1, 8, 9\}$, there are two valid $X = u$ correspondingly.

The final PMF is

$$p_Z(v) = \begin{cases} \frac{1}{12} & \text{if } 0 \leq v \leq 7 \\ \frac{1}{18} & \text{if } v \in \{-2, -1, 8, 9\} \\ \frac{1}{36} & \text{if } v \in \{-4, -3, 10, 11\} \\ 0 & \text{else} \end{cases}$$

3. **[Marginal and Conditional PMFs]**

Let the discrete random variables X and Y take values in $\{0, 1, 2, \dots\}$ and have the joint probability mass function (PMF) given by:

$$p(x, y) = \begin{cases} (1-\theta_1)^x (1-\theta_2)^y \cdot \theta_1 \theta_2, & \text{for } x, y \in \{0, 1, 2, \dots\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$.

Answer the following questions about X and Y.

(a) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

Solution: The marginal PMF of X is:

$$p_X(x) = \sum_{y=0}^{\infty} p(x, y) = (1-\theta_1)^x \cdot \theta_1 \cdot \sum_{y=0}^{\infty} (1-\theta_2)^y \cdot \theta_2$$

The sum is a geometric series of the form $\sum_{y=0}^{\infty} r^y = \frac{1}{1-r}$ for $|r| < 1$. Here, $r = 1 - \theta_2$, so:

$$\sum_{y=0}^{\infty} (1-\theta_2)^y \cdot \theta_2 = \theta_2 \cdot \sum_{y=0}^{\infty} (1-\theta_2)^y = \theta_2 \cdot \frac{1}{1-(1-\theta_2)} = \theta_2 \cdot \frac{1}{\theta_2} = 1$$

Therefore,

$$p_X(x) = (1-\theta_1)^x \cdot \theta_1$$

Similarly, the marginal PMF of Y is:

$$p_Y(y) = \sum_{x=0}^{\infty} p(x, y) = (1-\theta_2)^y \cdot \theta_2$$

(b) Find the conditional PMF $p_{Y|X}(y|x)$ and the conditional expectation $\mathbb{E}[Y|X = x]$.

Solution: The conditional PMF is given by:

$$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)} = \frac{(1-\theta_1)^x (1-\theta_2)^y \cdot \theta_1 \theta_2}{(1-\theta_1)^x \cdot \theta_1} = (1-\theta_2)^y \cdot \theta_2$$

So $Y|X = x \sim \text{Geometric}(\theta_2)$, independent of x.

The conditional expectation is:

$$\mathbb{E}[Y|X = x] = \sum_{y=0}^{\infty} y \cdot (1-\theta_2)^y \cdot \theta_2 = \frac{1-\theta_2}{\theta_2}$$

(c) Are X and Y independent? Justify your answer.

Solution: Yes, X and Y are independent because the joint PMF factors as:

$$p(x, y) = (1 - \theta_1)^x \cdot \theta_1 \cdot (1 - \theta_2)^y \cdot \theta_2 = p_X(x) \cdot p_Y(y)$$

This factorization confirms independence.

4. **[Hypothesis Testing]**
In a communication system, a bit $X \in \{0, 1\}$ is transmitted. The received signal is $Y = X+W$ where the random noise W is standard normal (i.e., Gaussian distribution with mean 0 and variance 1). The goal is to make a decision on whether X is 0 or 1 based on the received signal Y.

(a) Suppose X is equally likely to be 0 or 1. Find the MAP decision rule to decide if X is 0 or 1.

Solution: If $X = 0$ then $Y \sim \mathcal{N}(0, 1)$ and if $X = 1$ then $Y \sim \mathcal{N}(1, 1)$. Since the prior probabilities are equal, the MAP rule and ML rule are identical. Suppose we observe $Y = v$. The ML decision rule decides $X = 0$ when the likelihood (pdf value) of $Y = v$ conditioned on $X = 0$ is larger than the likelihood (pdf value) of $Y = v$ conditioned on $X = 1$. We do this by comparing the two conditional pdfs: decide $X = 0$ if

$$\frac{1}{\sqrt{2\pi}} e^{-v^2/2} > \frac{1}{\sqrt{2\pi}} e^{-(v-1)^2/2} \tag{1}$$

$$-\frac{v^2}{2} > -\frac{(v-1)^2}{2} \tag{2}$$

$$v < \frac{1}{2}. \tag{3}$$

The MAP rule is intuitive: decide $X = 0$ if $Y < \frac{1}{2}$ and $X = 1$ otherwise.

(b) Now suppose that $P[X = 0] = 0.8$ and $P[X = 1] = 0.2$. Find the MAP decision rule to decide if X is 0 or 1.

Solution: The MAP decision rule decides $X = 0$ when the weighted likelihood (pdf value) of $Y = v$ conditioned on $X = 0$ is larger than the weighted likelihood (pdf value) of $Y = v$ conditioned on $X = 1$; the weighting of the likelihoods is proportional to the prior probabilities of the two values of X. We discover the MAP rule by comparing the two weighted conditional pdfs: decide $X = 0$ if

$$0.8 \frac{1}{\sqrt{2\pi}} e^{-v^2/2} > 0.2 \frac{1}{\sqrt{2\pi}} e^{-(v-1)^2/2} \tag{4}$$

$$\ln(0.8) - \frac{v^2}{2} > \ln(0.2) - \frac{(v-1)^2}{2} \tag{5}$$

$$v < \ln(4) + 0.5 \tag{6}$$

The MAP rule is thus: decide $X = 0$ if $Y < \ln(4) + 0.5$ and $X = 1$ otherwise.